International Journal of Theoretical Physics, Vol. 33, No. 4, 1994

Difference Posets, Effects, and Quantum Measurements

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Received August 1, 1993

Difference posets as generalizations of quantum logics, orthoalgebras, and effects are studied. Observables and measures generalizing normalized POV-measures and generalized measures on sets of effects are introduced. Characterization of orthomodularity of subsets of a difference poset in terms of triangle closedness and regularity of these subsets enables us to characterize observables with a Boolean range. Boolean powers of difference posets are investigated; they have similar properties to that of tensor products, and their connection with quantum measurements is studied.

1. INTRODUCTION

For the mathematical foundations of quantum mechanics, a Hilbert space model plays, according to von Neumann (1932), an important role, and we recall that Mackey (1963, Axiom VII and p. 73) assumed that the set of all quantum mechanical events is isomorphic to the space of all closed subspaces, $L(H)$, of a separable, complex, infinite-dimensional Hilbert space H. In the historic paper of Birkhoff and von Neumann (1936), the notion of *quantum logics* was introduced. Now quantum logics generalize both models of classical mechanics, *Boolean algebras* [= Kolmogorov (1933) model] and that of Hilbert space quantum mechanics.

If L describes a propositional system of a physical system, then an observable is represented by a suitable morphism x from some σ-algebra $\mathscr{S}$ of subsets of a nonvoid set X into L; preferably we use $X = \mathbb{R}$ and $\mathscr{S}$ is the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ of the real line $\mathbb{R}$, or $X = \mathbb{R}^n$ and $\mathscr{S} = \mathscr{B}(\mathbb{R}^n)$, and a state of the system is characterized by an additive, σ-additive or completely

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additive real-valued mapping on L. This in classical mechanics leads to measurable functions (=random variables) and in the Hilbert space model to self-adjoint operators, and probability measures or Gleason measures, respectively.

More general structures, *orthoalgebras*, have been introduced by Randall and Foulis (1979, 1981), and they enable one to introduce a tensor product of orthoalgebras (Foulis and Bennett, n.d.), which is an important tool for coupled systems.

Events of quantum logics or orthoalgebras have a "yes–no" character and therefore they do not describe unsharp measurements. The effort to include them leads in Hilbert space quantum mechanics to the set of all *effects* (Busch *et al.*, 1991), i.e., to the set $\mathscr{E}(H)$ of all Hermitian operators with spectra in the interval [0, 1]. Then "yes–no" events, i.e., those having spectrum in the two-point set $\{0, 1\}$, correspond to orthogonal projection operators on H, and $L(H) \subset \mathscr{E}(H)$, and an unsharp measurement is represented by a POV-*measure*.

Recently there has appeared a new mathematical model, *difference posets* (or D-posets, for short), introduced by Kôpka and Chovanec (1994), which generalizes quantum logics and orthoalgebras as well as the set of effects, and which was inspired by an investigation of the possibility of introducing fuzzy set ideas to quantum structures models (Kôpka, 1992). In this model, the difference operation is a primary notion from which we derive other, usual notions important for measurements.

The aim of the present paper is the investigation of difference posets from the point of view of applications for quantum measurements. We study observables and states, and show their connection with POV-measures in the set of effects. Conditions in order that observables have a Boolean range will be presented; this part generalizes results by Lahti and Maczyński (1992). Finally, we study Boolean powers of difference posets which can model a coupled system consisting of a microscopic unsharp quantum structure and a measuring apparatus.

2. DIFFERENCE POSETS

A D-*poset*, or a *difference poset*, is a partially ordered set L with a partial ordering $\le$, greatest element 1, and partial binary operation $\ominus: L \times L \to L$, called a *difference*, such that, for $a, b \in L$, $b \ominus a$ is defined if and only if $a \le b$; the following axioms hold for $a, b, c \in L$:

(DPi) $b \ominus a \le b$.

(DPii) $b \ominus (b \ominus a) = a$.

(DPiii) $a \le b \le c \Rightarrow c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The following statements have been provided in Kôpka and Chovanec (1994):

Proposition 2.1. Let a, b, c, d be elements of a D-poset L. Then:

(i) $1 \ominus 1$ is the least element of L; denote it by 0.
(ii) $a \ominus 0 = a$.
(iii) $a \ominus a = 0$.
(iv) $a \le b \Rightarrow b \ominus a = 0 \Leftrightarrow b = a$.
(v) $a \le b \Rightarrow b \ominus a = b \Leftrightarrow a = 0$.
(vi) $a \le b \le c \Rightarrow b \ominus a \le c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.
(vii) $b \le c$, $a \le c \ominus b \Rightarrow b \le c \ominus a$ and $(c \ominus b) \ominus a = (c \ominus a) \ominus b$.
(viii) $a \le b \le c \Rightarrow a \le c \ominus (b \ominus a)$ and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

Remark 2.2 (Navara and Pták, n.d.). A poset L with least and greatest elements 0 and 1, respectively, and with a partial binary operation $\ominus: L \times L \to L$, where $b \ominus a$ is defined iff $a < b$, such that for $a, b, c \in L$ we have

(i) $a \ominus 0 = a$
(ii) if $a \le b \le c$, then $c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$ is a D-poset.

For any element $a \in L$ we put

$$a^\perp := 1 \ominus a$$

Then (i) $a^{\perp\perp} = a$; (ii) $a \le b$ implies $b^\perp \le a^\perp$. Two elements a and b of L are *orthogonal*, and we write $a \perp b$, iff $a \le b^\perp$ (iff $b \le a^\perp$).

Now we introduce a binary operation $\oplus: L \times L \to L$ such that an element $c = a \oplus b$ in L is defined iff $a \perp b$, and for c we have $b \le c$ and $a = c \ominus b$. The partial operation $\oplus$ is defined correctly because if there exists $c_1 \in L$ with $b \le c_1$ and $a = c_1 \ominus b$, then, by (viii) of Proposition 2.1 and (DPii), we have

$$(1 \ominus (c \ominus b)) \ominus b = 1 \ominus c = (1 \ominus (c_1 \ominus b)) \ominus b = 1 \ominus c_1$$

which implies $c = c_1$. Moreover,

$$a \oplus b = (a^\perp \ominus b)^\perp = (b^\perp \ominus a)^\perp \tag{2.1}$$

Indeed, denote by $x = (a^\perp \ominus b)^\perp$. From (vii) of Proposition 2.1, we conclude that $x = (b^\perp \ominus a)^\perp$. Therefore, $x^\perp = a^\perp \ominus b$, which means $a \le x$, analogously, $b \le x$. Calculate $x \ominus a = (1 \ominus (b^\perp \ominus a)) \ominus a = 1 \ominus b^\perp = b$ when we have used (viii) of Proposition 2.1.

The operation $\oplus$ is commutative (this is evident) and associative: suppose that $y = a \oplus b$ and $z = (a \oplus b) \oplus c$ exist in L. By (DPiii) we have

$$(z \ominus a) \ominus (z \ominus y) = y \ominus a$$

$$(z \ominus a) \ominus c = b$$

$$z \ominus a = b \oplus c \in L$$

$$z = a \oplus (b \oplus c) \in L$$

Very important examples of difference posets are orthomodular posets (=quantum logics), orthoalgebras, and sets of effects.

3. ORTHOMODULAR POSETS

An *orthomodular poset* (OMP) is a partially ordered set L with an ordering $\leq$, the minimal and maximal elements 0 and 1, respectively, and an orthocomplementation $\perp: L \to L$ such that

(OMi) $a^{\perp\perp} = a$ for any $a \in L$.
(OMii) $a \vee a^{\perp} = 1$ for any $a \in L$.
(OMiii) If $a \leq b$, then $b^{\perp} \leq a^{\perp}$.
(OMiv) If $a \leq b^{\perp}$ (and we write $a \perp b$), then $a \vee b \in L$.
(OMv) If $a \leq b$, then $b = a \vee (a \vee b^{\perp})^{\perp}$ (orthomodular law).

If in an orthomodular poset L the join of any sequence (any system) of mutually orthogonal elements exists, we say that L is a *σ-orthomodular poset* (a *complete orthomodular poset*). An *orthomodular lattice* is an orthomodular poset L such that, for any a, $b \in L$, $a \vee b$ exists in L (using the de Morgan laws, $a \wedge b$ exists in L, too). A distributive orthomodular lattice is called a *Boolean algebra*. We recall that an orthomodular lattice L is a Boolean algebra iff for any pair a, $b \in L$ there are three mutually orthogonal elements a_1, b_1, $c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. For more details concerning orthomodular posets and lattices see, e.g., Kalmbach (1983) and Pták and Pulmannová (1991).

One of the most important cases of orthomodular lattices is the system of all closed subspaces, $L(H)$, of a real or complex Hilbert space H, with an inner product $(\cdot, \cdot)$. Here the partial ordering $\leq$ is induced by the natural set-theoretic inclusion, and $M^{\perp} = \{x \in H: (x, y) = 0$ for any $y \in M\}$. Then $L(H)$ is a complete orthomodular lattice, which is not a Boolean algebra, if $\dim H \neq 1$. This structure plays a crucial role in axiomatic foundations of quantum mechanics.

If for two elements a, b of an OMP L, with $a \leq b$, we define by (OMv)

$$b \ominus a := (a \vee b^{\perp})^{\perp}$$

then L with $\leq$, 1, and $\ominus$ is a difference poset.

4. ORTHOALGEBRAS

An *orthoalgebra* is a set L with two particular elements 0, 1, and with a partial binary operation $\oplus: L \times L \to L$ such that for all $a, b, c \in L$ we have:

(OAi) If $a \oplus b \in L$, then $b \oplus a \in L$ and $a \oplus b = b \oplus a$ (commutativity).
(OAii) If $b \oplus c \in L$ and $a \oplus (b \oplus c) \in L$, then $a \oplus b \in L$ and $(a \oplus b) \oplus c \in L$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).
(OAiii) for any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined, and $a \oplus b = 1$ (orthocomplementation).
(OAiv) If $a \oplus a$ is defined, then $a = 0$ (consistency).

If the assumptions of (ii) are satisfied, we write $a \oplus b \oplus c$ for the element $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ in L.

Let a and b be two elements of an orthoalgebra L. We say that (i) a is *orthogonal* to b and write $a \perp b$ iff $a \oplus b$ is defined in L; (ii) a is *less or equal* b and write $a \leq b$ iff there exists an element $c \in L$ such that $a \perp c$ and $a \oplus c = b$ (in this case we also write $b \geq a$); (iii) b is the *orthocomplement* of a iff b is a (unique) element of L such that $b \perp a$ and $a \oplus b = 1$, and it is written as $a^\perp$.

If $a \leq b$, for the element c in (ii) with $a \oplus c = b$ we write $c = b \ominus a$, and c is called the *difference* of a in b. It is evident that

$$b \ominus a = (a \oplus b^\perp)^\perp \tag{4.1}$$

In Foulis *et al.* (1992) there are proofs of the following statements:

Proposition 4.1. Let a, b, and c be elements of an orthoalgebra L. Then:

(i) $a \perp b \Leftrightarrow b \perp a$.
(ii) $a \perp a \Rightarrow a = 0$.
(iii) $a \perp 1 \Leftrightarrow a = 0$.
(iv) $a^{\perp\perp} = a$.
(v) $1^\perp = 0$ and $0^\perp = 1$.
(vi) $a \perp b \Rightarrow a \perp (a \oplus b)^\perp$, $a \oplus (a \oplus b)^\perp = b^\perp$.
(vii) $a \perp b \Leftrightarrow a \leq b^\perp$.
(viii) $a \leq b \Rightarrow b = a \oplus (a \oplus b^\perp)^\perp$.
(ix) $a \oplus b = a \oplus c \Rightarrow b = c$.
(x) $a \oplus b \leq a \oplus c \Rightarrow b \leq c$.
(xi) $0 \leq a \leq 1$, and $\leq$ is a partial ordering on L.
(xii) $a \leq b \Rightarrow b^\perp \leq a^\perp$.
(xiii) $a \wedge a^\perp = 0$, $a \vee a^\perp = 1$.

(xiv) $a \perp b,\ a \vee b \in L \Rightarrow a \oplus b = a \vee b$.
(xv) $a^{\perp} = 1 \ominus a$.
(xvi) $a \le b \Leftrightarrow b = a \oplus (b \ominus a)$.
(xvii) $a = a \ominus 0$.
(xviii) $a \le b \le c \Leftrightarrow (c \ominus b) \oplus (b \ominus a) = c \ominus a$.
(xix) $a \le b \le c \Leftrightarrow (c \ominus a) \ominus (c \ominus b) = b \ominus a$.

We see that if L is an orthomodular poset and $a \oplus b := a \vee b$ whenever $a \perp b$ in L, then L with 0, 1, $\oplus$ is an orthoalgebra. The converse statement does not hold, in general, as it follows from the example of R. Wright (Foulis *et al.*, 1992):

Example 4.2. Let $L = \{0, 1, a, b, c, e, f, a^{\perp}, b^{\perp}, c^{\perp}, d^{\perp}, e^{\perp}, f^{\perp}\}$ with $a \oplus b = d \oplus e = c^{\perp}$, $b \oplus c = e \oplus f = a^{\perp}$, $c \oplus d = f \oplus a = e^{\perp}$, $c \oplus e = d^{\perp}$, $a \oplus c = b^{\perp}$, $e \oplus a = f^{\perp}$ is an orthoalgebra that is not an orthomodular poset.

We recall that an orthoalgebra L is an OMP iff $a \perp b$ implies $a \vee b \in L$.

It is evident that any orthoalgebra L is a D-poset when a difference $\ominus$ is defined by (4.1). Indeed, (DPi) and (DPii) are trivially satisfied, and (DPiii) follows from (xix) of Proposition 4.1.

By Navara and Pták (n.d.), we conclude that a D-poset L with 0, 1, and $\oplus$, defined by (2.1), is an orthoalgebra if and only if $a \le 1 \ominus a$ implies $a = 0$. Therefore, it is not hard to give many examples of D-posets which are not orthoalgebras; such ones are sets of effects.

5. SETS OF EFFECTS

The general line of this section is that the description of a physical system is based on a probabilistic duality of states and effects (Ludwig, 1983). It means the approach where the set of states of a physical system is represented as the base K of a base-norm Banach space (E, K).

We recall that E is supposed to be a real linear space and K is *base*, i.e., a convex subset of E such that $x, y \in K$, $s, t \ge 0$ with $sx = ty$ imply $s = t$, and $\mathrm{lin}(K) = E$, where lin means the linear span over given set. Then there is a unique linear functional $e: E \to \mathbb{R}$ such that $e(x) = 1$ for any $x \in K$. For this *Minkowski functional* e we have, for any $x \in E$, $e(x) = \inf\{t \ge 0: x \in t\,\mathrm{con}(K \cup -K)\}$, where con means the convex hull. Putting $\|x\| = \|x\|_K := e(x)$, $\|\cdot\|$ is a semi norm on E which is additive on K. Supposing that $\|\cdot\|$ is a norm on E, we call (E, K) a *base-norm space*. Providing E is a complete normed space with respect to $\|\cdot\|$, (E, K) is called a *base-norm Banach space* [for more details, see, e.g., Alfsen (1972)].

Let E^* denote the Banach dual space corresponding to the base-norm Banach space (E, K). We write, for two linear functionals $f, g \in E^*$, $f \leq g$ iff $f(x) \leq g(x)$, $x \in E$. Since K is generating for E, we conclude that $f \leq g$ iff $f(x) \leq g(x)$, $x \in K$. The Minkowski functional e belongs to E^*, and e is an *order-unit* for E^*, i.e., for any $f \in E^*$, there is an integer n such that $-ne \leq f \leq ne$, and $(E^*, \leq, e)$ is called an *order-unit space* with the sup-norm $\|f\| = \sup_{\|x\| \leq 1} |f(x)| = \sup_{x \in K} |f(x)|, f \in E^*$.

An *effect* is an element of the order-unit interval $\mathscr{E} = \mathscr{E}(E) := \{f \in E^*: o \leq f \leq e\}$, where o is the zero functional. The set of all effects $\mathscr{E}(E)$ is a poset with the minimal and maximal elements o and e, respectively, and weak *-compact. The set of all its extreme points, $\mathrm{Ext}(\mathscr{E})$, has the following properties: if $f \in \mathrm{Ext}(\mathscr{E})$, then $e - f \in \mathrm{Ext}(\mathscr{E})$, the following join and meet exist in $\mathrm{Ext}(\mathscr{E})$, and $f \vee (e - f) = e$, $f \wedge (e - f) = o$, and, according to the Krein–Milman theorem, any element of $\mathscr{E}$ can be weak *-approximated by finite convex combinations of elements of $\mathrm{Ext}(\mathscr{E})$.

An *σ-observable* is a mapping $A: \mathscr{B}(\mathbb{R}) \to \mathscr{E}(E)$ such that $A(\emptyset) = o$, $A(\mathbb{R}) = e$, and $A(\bigcup_{i=1}^{\infty} X_i) = \sum_{i=1}^{\infty} A(X_i)$ for any sequence of mutually disjoint sets $X_i \in \mathscr{B}(\mathbb{R})$ (with the sum converging in the weak *-topology of E^*).

A mapping μ from the set $\mathscr{E}(E)$ into the real interval $[0, 1]$ such that (i) $\mu(e) = 1$ and (ii) if $\sum_{i \in I} f_i \leq e$ (in the weak *-topology of E^*), then $\mu(\sum_{i \in I} f_i) = \sum_{i \in I} \mu(f_i)$ is said to be a *finitely additive*, *σ-additive*, or *completely additive state*, respectively, whenever the index set I in (ii) is always finite, countable, or arbitrary.

Remark 5.1. It is easy to see that if for two effects $f, g \in \mathscr{E}(E)$ with $f \leq g$, we define $g \ominus f := g - f$, then $\mathscr{E}(E)$ with $\leq$, e, and $\ominus$ is a difference poset.

Remark 5.2. The Hilbert space quantum mechanics is an important example of the above general framework of base-norm Banach spaces. Here we put $E = \mathrm{Tr}(H)$, the set of all Hermitian trace class operators on H, and K is the set of all von Neumann operators on H. Then the dual E^* can be identified with the set of all Hermitian operators on H, and e corresponds now to the identical operator I on H. The set of all effects, $\mathscr{E}(H)$, is thus the set of all Hermitian operators between O and I, and the set of all extreme effects corresponds to the set of all orthogonal projection operators on H. In addition, any σ-observable on $\mathscr{E}(H)$ corresponds to a so-called POV-measure (Busch *et al.*, 1991), and if $\dim H \neq 2$, any completely additive state μ on $\mathscr{E}(H)$ is in a one-to-one correspondence with a von Neumann operator T on H (Busch *et al.*, 1991; Dvurečenskij, 1993) via

$$\mu(P) = \mathrm{tr}(TP), \qquad P \in \mathscr{E}(H) \tag{5.1}$$

It is worth noting that if P, Q are two orthogonal projectors on H, then $P \wedge Q$ and $P \vee Q$ exist in $L(H)$ as well as in $\mathscr{E}(H)$,[2] and both meets and joins are identical. Indeed, let B be a positive Hermitian operator on H such that $B \leq P, Q$. Let M and N be ranges of P and Q, respectively, and let $C = B^{1/2}$. Since $\|Cx\|^2 \leq \|Px\|^2$, C and hence B vanish on $M^\perp$. Similarly B vanishes on $N^\perp$ and therefore $B = 0$ on $(M \cap N)^\perp$, which means that B is invariant on $M \cap N$, and for $x \in M \cap N$, $(Bx, x) \leq \|x\|^2$, so that $B \leq R := P \wedge Q$, where R is the orthogonal projector onto $M \cap N$.

The assertion for join follows from the de Morgan law. Moreover, $\mathscr{E}(H)$ is not a lattice if $\dim H > 1$ (Cattaneo and Nisticó, 1989).

In Sections 6 and 7, we show that notions of σ-observables and states coincide with those for general difference posets.

6. MEASURES ON DIFFERENCE POSETS

The general notions of group-valued measures on D-posets have been introduced in Dvurečenskij and Riečan (1994). For our aims we introduce the following notions.

Let $F = \{a_1, \dots, a_n\}$ be a finite sequence in L. Recursively we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n \tag{6.1}$$

supposing that $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in D-posets we conclude that (6.1) is correctly defined. Definitorically we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$ and any k with $1 \leq k \leq n$ we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n} \tag{6.2}$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n) \tag{6.3}$$

We say that a finite sequence $F = \{a_1, \dots, a_n\}$ of L is $\bigoplus$*-orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case we say that F has a $\bigoplus$*-sum*, $\bigoplus_{i=1}^n a_i$, defined via

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n \tag{6.4}$$

It is clear that two elements a and b of L are orthogonal, i.e., $a \perp b$, iff $\{a, b\}$ is $\bigoplus$-orthogonal.

An arbitrary system $G = \{a_i\}_{i \in I}$ of not necessarily different elements of L is $\bigoplus$*-orthogonal* iff, for every finite subset F of I, the system $\{a_i\}_{i \in F}$

[2] $L(H)$ can be identified with the $P(H)$ of all orthogonal projectors on H.

is $\oplus$-orthogonal. If $G = \{a_i\}_{i\in I}$ is $\oplus$-orthogonal, so is any $\{a_i\}_{i\in J}$ for any $J \subseteq I$. An $\oplus$-orthogonal system $G = \{a_i\}_{i\in I}$ of L has a $\oplus$*-sum* in L, written as $\bigoplus_{i\in I} a_i$, iff in L there exists the join

$$\bigoplus_{i\in I} a_i := \bigvee_F \bigoplus_{i\in F} a_i \tag{6.5}$$

where F runs over all finite subsets in I. In this case, we also write $\bigoplus G := \bigoplus_{i\in I} a_i$.

It is evident that if $G = \{a_1, \ldots, a_n\}$ is $\oplus$-orthogonal, then the $\oplus$-sums defined by (6.4) and (6.5) coincide.

We say that a D-poset L is a *complete D-poset* (*σ-D-poset*) if, for any $\oplus$-orthogonal system (any countable $\oplus$-orthogonal sequence) G of L, there exists the $\oplus$-sum in L. It is straightforward to verify that a D-poset L is a D-σ-poset if, for any sequence $\{a_i\}$ in L with $a_1 \le a_2 \le \cdots$, the join $\bigvee_{i=1}^{\infty} a_i$ exists in L.

A mapping $\mu: L \to [0, 1]$ such that (i) $\mu(1) = 1$ and (ii) $\mu(\bigoplus_{i\in I} a_i) = \sum_{i\in I} \mu(a_i)$ if $\{a_i\}_{i\in I}$ is $\oplus$-orthogonal with the $\oplus$-sum $\bigoplus_{i\in I} a_i$ in L, is said to be a *finitely additive*, *σ-additive*, or *completely additive state*, respectively, whenever (ii) holds for any finite, countable, or arbitrary index set I.

It is evident that a mapping $\mu: L \to [0, 1]$ is a finitely additive state on L iff (i) $\mu(1) = 1$, (ii) $\mu(b \ominus a) = \mu(b) - \mu(a)$ if $a \le b$, and if, in addition, (iii) $a_n \nearrow a$ (i.e., $a_1 \le a_1 \le \cdots$, $\bigvee_n a_n = a$), then $\mu(a_n) \nearrow \mu(a)$, then μ is σ-additive.

We recall that it is not hard to verify that these notions of states coincide with the usual ones for OMPs, orthoalgebras, and effects.

7. OBSERVABLES IN DIFFERENCE POSETS

A mapping $x: \mathscr{B}(\mathrm{R})$ into a D-poset L such that (i) $x(\mathbb{R}) = 1$ and (ii) $x(E) \perp x(F)$ whenever $E \cap F = \emptyset$, $E, F \in \mathscr{B}(\mathbb{R})$, and $x(E \cup F) = x(E) \oplus x(F)$, is said to be an *observable*.[3] It is easy to see that $x(\emptyset) = 0$, $x(\mathbb{R}\setminus E) = x(E)^{\perp}$, $E \in \mathscr{B}(\mathbb{R})$, and $x(\bigcup_{i=1}^{n} E_i) = \bigoplus_{i=1}^{n} x(E_i)$ whenever $E_i \cap E_j = \emptyset$ for $1 \le i < j \le n$.

If (ii) in the definition of observables is changed to the requirement (ii)* $\{x(E_i)\}_{i=1}^{\infty}$ is $\oplus$-orthogonal whenever $\{E_i\}$ is a sequence of mutually disjoint subsets from $\mathscr{B}(\mathbb{R})$, and $x(\bigcup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} x(E_i)$, we say that x is a *σ-observable*. It is straightforward to see that previous notions of σ-observables for effects coincide with ours. Moreover, a map $x: \mathscr{B}(\mathbb{R}) \to L$

[3]More generally, $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ can be changed to a measurable space $(\Omega, \mathscr{F})$, and $(\Omega, \mathscr{F})$ is said to be a *value space* of the observable x.

is a σ-observable iff (i) $x(\mathbb{R}) = 1$, (ii) if $E \subseteq F$, then $x(E) \leq x(F)$ and $x(F \setminus E) = x(F) \ominus x(E)$, and (iii) if $E_n \nearrow E$, then $x(E_n) \nearrow x(E)$ (Kôpka and Chovanec, 1994).

In addition, if μ is a σ-additive state on a D-σ-poset L and x is a σ-observable of L, then $\mu_x: E \mapsto \mu(x(E))$, $E \in \mathscr{B}(\mathbb{R})$, is a usual probability measure on $\mathscr{B}(\mathbb{R})$, and via $\mu(x) = \int_{\mathbb{R}} t\, d\mu_x(t)$ we can define a *mean value* of x in μ supposing that the latter integral exists and is finite.

To give some examples of observables, we present the following notation and propositions.

Let, for any $t \in T$, $A_t = \{a_t^i\}_{i \in I_t}$ be a system of not necessarily different elements in L, such that $I_t \cap I_s = \emptyset$ for $t \neq s$, $t, s \in T$. Then, for $A = \{a_t^i\}_{i \in I_t, t \in T}$ we shall write $A := \dot{\bigcup}_{t \in T} A_t$.

Proposition 7.1. If $A = \{a_i\}_{i \in I}$ and $B = \{b_j\}_{j \in J}$, with $I \cap J = \emptyset$, are two systems of a D-poset L such that $A \dot{\cup} B$ is $\bigoplus$-orthogonal and $\bigoplus A$ and $\bigoplus B$ exist in L, then $\bigoplus (A \dot{\cup} B)$ exists in L, and

$$\bigoplus (A \dot{\cup} B) = \bigoplus A \oplus \bigoplus B \tag{7.1}$$

Proof. First we show that $\bigoplus A \perp \bigoplus B$. Indeed, since for any n, m we have $\bigoplus_{i=1}^n a_i \oplus \bigoplus_{j=1}^m b_j \in L$, we have $\bigoplus_{i=1}^n a_i \leq (\bigoplus_{j=1}^m b_j)^\perp$, so that $\bigoplus A \leq (\bigoplus_{j=1}^m b_j)^\perp$, which gives $\bigoplus_{j=1}^m b_j \leq (\bigoplus A)^\perp$, so that $\bigoplus B \perp \bigoplus A$.

It is clear that $\bigoplus_{i=1}^n a_i \oplus \bigoplus_{j=1}^m b_j \leq \bigoplus A \oplus \bigoplus B$. Now let c be an arbitrary element of L such that $\bigoplus_{i=1}^n a_i \oplus \bigoplus_{j=1}^m b_j \leq c$ for any n, m. Then $\bigoplus_{i=1}^n a_i \leq c \ominus \bigoplus_{j=1}^m b_j$ and $\bigoplus A \leq c \ominus \bigoplus_{j=1}^m b_j$. Therefore, $\bigoplus_{j=1}^m b_j \leq c \ominus \bigoplus A$, which gives $\bigoplus A \oplus \bigoplus B \leq c$, so that (7.1) holds. ■

Proposition 7.2. Let $A = \dot{\bigcup}_{t \in T} A_t$, where $A_t = \{a_t^i\}_{i \in I_t}$, $I_t \cap I_s = \emptyset$ for $t \neq s$, $t, s \in T$, to be an $\bigoplus$-orthogonal system of a D-poset L and let, for any $t \in T$, $\bigoplus A_t$ exist in L, and $\bigoplus_{t \in T}(\bigoplus A_t) \in L$. Then $\bigoplus A$ exists in L, and

$$\bigoplus A = \bigoplus_{t \in T} (\bigoplus A_t) \tag{7.2}$$

Proof. Let $\{a_1, \ldots, a_n\}$ be a finite sequence from A, and $a_i \in A_{t_i}$ for any $i = 1, \ldots, n$. Then $\bigoplus_{i=1}^n a_i \leq \bigoplus_{i=1}^n (\bigoplus A_{t_i})$, so that $\bigoplus_{i=1}^n a_i \leq \bigoplus_{t \in T}(\bigoplus A_t)$. Suppose now that $\bigoplus_{i=1}^n a_i \leq c$ for all $a_i \in A$ and any n. Then, for any $A_{t_1}, \ldots, A_{t_m}$, we have by Proposition 7.1, $\bigoplus_{k=1}^m (\bigoplus A_{t_k}) = \bigoplus (\dot{\bigcup}_{k=1}^m A_{t_k})$, so that $\bigoplus_{k=1}^m (\bigoplus A_{t_k}) \leq c$, which means $\bigoplus_{t \in T}(\bigoplus A_t) \leq c$, and consequently, (7.2) holds. ■

Corollary 7.3. Let L be a D-σ-*poset* and $\{a_n\}_{n=0}^\infty$ be a $\bigoplus$-orthogonal sequence of elements in L such that $\bigoplus_{n=0}^\infty a_n = 1$. Then the mapping x

defined via

$$x(E) = \bigoplus \{a_i\}_{i\in E}, \qquad E\in\mathscr{B}(\mathbb{R}) \tag{7.3}$$

is a σ-observable of L.

Proof. It follows from Propositions 7.1. and 7.2. ■

In particular, given an element a of a D-poset L, let $a_0 = a^\perp$, $a_1 = a$. Then x_a defined by (7.3) is a so-called *question observable* (i.e., an analog of characteristic functions).

Let A be a subset of D-poset L such that (i) $1\in A$, (ii) if $a\in A$, then $a^\perp\in A$, (iii) if $a, b\in A$, $a \perp b$, then $a\oplus b\in A$, (iv) for any $a, b\in A$, $a \vee_A b\in A$ [(iv*) for $\{a_i\}_{i=1}^\infty$ from A, $\bigvee_{i=1A}^\infty a_i\in A$], where the join $\bigvee_A$ is taken in A (not in L), and (v) A with respect to 0, 1 $\perp$, and $\vee_A$ is a Boolean (σ-) algebra [for the definition of Boolean algebras, see, e.g., Sikorski (1964) is said to be a *Boolean algebra* (*Boolean σ-algebra*) of L.

If A is a Boolean algebra of L, then $a\oplus b = a \vee_A b$ whenever $a, b\in A$. We recall that there are examples of D-subposets[4] A of L (see below) such that $a\oplus b\in A$ and $a \vee_A b \neq a\oplus b$.

Let x be an observable of L; then by a *range* of x we mean the set $\mathscr{R} := \mathscr{R}(x) = \{x(E): E\in\mathscr{B}(\mathbb{R})\}$. If L is an OMP or an orthoalgebra, the range $\mathscr{R}(x)$ is always a Boolean algebra of L (Varadarajan, 1985; Pták and Pulmannová, 1991; Pulmannová, 1993*a*). In difference posets this statement does not hold, in general. Indeed, let $L = \mathscr{E}(H)$, and $a = \frac{1}{2}I$. Then $a^\perp = \frac{1}{2}I$ and the range of the question observable x_a is $\{O, \frac{1}{2}I, I\}$, but $\frac{1}{2}I \vee_{\mathscr{R}} \frac{1}{2}I = \frac{1}{2}I \neq \frac{1}{2}I\oplus\frac{1}{2}I = I$.

The Boolean character of the range of an observable is an important feature of observables, and in the frame of effects this problem has been solved in Lahti and Maczyński (1992). We now present a more general solution for difference posets.

We say that an observable x of a D-poset L is *regular* if $x(E) \leq x(\mathbb{R}\setminus E)$ for some $E\in\mathscr{B}(\mathbb{R})$ implies $x(E) = 0$.

Theorem 7.4. An observable (σ-observable) x of a difference poset L is regular if and only if the range $\mathscr{R}(x)$ is a Boolean algebra (Boolean σ-algebra) of L. Then

$$x(E) = \bigvee_{i=1}^{\infty}{}_{\mathscr{R}}\, x(E_i), \qquad x(F) = \bigwedge_{i=1}^{\infty}{}_{\mathscr{R}}\, x(F_i) \tag{7.4}$$

whenever $E = \bigcup_{i=1}^\infty E_i$, $F = \bigcap_{i=1}^\infty F_i$, $E_i, F_i\in\mathscr{B}(\mathbb{R})$.

[4]A subset A of a D-poset L is a D-*subposet* if (i) $1\in A$; (ii) if $a, b\in A$, $a\leq b$, then $b\ominus a\in A$.

Proof. Suppose that $\mathscr{R}(x)$ is a Boolean algebra of L and let $x(E) \leq x(E^c)$. Then $1 = x(\mathbb{R}) = x(E \cup E^c) = x(E) \vee_{\mathscr{R}} x(E^c) = x(E^c)$, thus $x(E) = 0$.

Conversely, let x be a regular observable. It is easy to see that $0, 1 \in \mathscr{R}(x)$ and if $x(E) \in \mathscr{R}(x)$, then $x(E)^{\perp} \in \mathscr{R}(x)$. Now we proceed by steps.

(i) Let $x(G) \leq x(F), x(F^c)$. Then $x(F), x(F^c) \leq x(G^c)$, so that $x(G) \leq x(F) \leq x(G^c)$, which yields $x(G) = 0$.

(ii) Let $x(E), x(F)$ be orthogonal elements of $\mathscr{R}(x)$. Then $x(E \cap F) \leq x(E)$, $x(E \cap F) \leq x(F^c), x(F)$, which by (i) gives $x(E \cap F) = 0$. Therefore, $x(E) = x(E \backslash F)$, $x(F) = x(F \backslash E)$, and $x(E) \oplus x(F) = x(E \backslash F \cup F \backslash E) = x(E \cup F) \in \mathscr{R}(x)$.

(iii) Let $\{x(E_1), \ldots, x(E_n)\}$ be $\bigoplus$-orthogonal. Then for $F_i = E_i \backslash \bigcup_{j \neq i} E_j$, $i = 1, \ldots, n$, we have $x(E_i) = x(F_i)$, and

$$x\left(\bigcup_{i=1}^{n} E_i\right) = x\left(\bigcup_{i=1}^{n} F_i\right) = \bigoplus_{i=1}^{n} x(F_i) = \bigoplus_{i=1}^{n} x(E_i) \in \mathscr{R}(x)$$

(iv) $\{x(E_1), \ldots, x(E_n)\}$ is $\bigoplus$-orthogonal iff $x(E_i) \perp x(E_j)$ for $i \neq j$. Indeed, this follows from the fact that for F_i from (iii) we have $x(E_i) = x(F_i)$, and

$$x\left(\bigcup_{i=1}^{n} E_i\right) = x\left(\bigcup_{i=1}^{n} F_i\right) = \bigoplus_{i=1}^{n} x(F_i) = \bigoplus_{i=1}^{n} x(E_i) \in \mathscr{R}(x)$$

(v) If $\{x(E_1), \ldots, x(E_n)\}$ is $\bigoplus$-orthogonal, then

$$x\left(\bigcup_{i=1}^{n} E_i\right) = \bigvee_{i=1}^{n}{}_{\mathscr{R}}\, x(E_i)$$

Indeed, if $E = \bigcup_{i=1}^{n} E_i$, then $x(E) \geq x(E_i)$, $i = 1, \ldots, n$. Let $x(G) \geq x(E_i)$, $i = 1, \ldots, n$. Then $\{x(G^c), x(E_1), \ldots, x(E_n)\}$ is, by (iv), $\bigoplus$-orthogonal, and therefore we have

$$\mathscr{R}(x) \ni x(G^c) \oplus \bigoplus_{i=1}^{n} x(E_i) = x(G^c) \oplus x(E)$$

which means $x(E) \leq x(G^c)^{\perp} = x(G)$.

(vi) Let $E = \bigcup_{i=1}^{n} E_i$; then

$$x(E) = \bigvee_{i=1}^{n}{}_{\mathscr{R}}\, x(E_i)$$

Indeed, every E_k is of the form

$$E_k = \bigcup_{j_1 \cdots j_n = 0}^{1} \bigcap_{i=1}^{n} E_i^{j_i}, \quad \text{where} \quad E_i^0 = E_i^c, \quad E_i^1 = E_i$$

Then by (v),

$$x(E) = \bigvee_{s\bullet}^{\mathcal{R}} x(F_s)$$

where F_s is of the form $\bigcap_{i=1}^{n} E_i^{j_i}$ for some suitable numbers $j_1, \ldots, i_n \in \{0, 1\}$.

If $x(G) \geq x(E_i)$, $i = 1, \ldots, n$, then $x(G) \geq x(F_s)$, and by (v) $x(G) \geq x(E)$.

(vii) Let $E = \bigcup_{n=1}^{\infty} E_n$ and define $G_n = \bigcup_{i=1}^{n} E_i$, $F_n = E_i \setminus \bigcup_{i=1}^{n-1} E_i$. Then $E = \bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} F_n$, $G_n = \bigcup_{i=1}^{n} F_i$. Calculate

$$x(E) = \bigoplus_{n=1}^{\infty} x(F_n) = \bigvee_{n=1}^{\infty} \bigoplus_{i=1}^{n} x(F_i) = \bigvee_{n=1}^{\infty} x(G_n) = \bigvee_{n=1}^{\infty} x\left(\bigcup_{i=1}^{n} E_i\right)$$

so that if $x(G) \geq x(E_n)$, $n \geq 1$, then by (vi), $x(G) \geq x(\bigcup_{i=1}^{n} E_i)$, which yields $x(G) \geq x(E)$. Thus we have proved the first part of (7.4).

Dually we prove the second part of (7.4). This distributivity of $\mathcal{R}(x)$ follows from the following: let $A, B, C \in \mathcal{B}(\mathbb{R})$ be given; then $x(A) \wedge (x(B) \vee x(C)) = x(A) \wedge (x(B \cup C)) = x(A \cap (B \cup C)) = x(A \cap B) \vee x(B \cap C) = x(A) \wedge x(B) \vee x(B) \wedge x(C)$, and this proves that $\mathcal{R}(x)$ is a Boolean (σ-) algebra of L. ■

Remark 7.5. We recall that in any D-poset L there exist at least two regular observables, namely the question observables x_1 and x_0 corresponding to 1 and 0. On the other hand, there are D-posets which possess no regular observables with the exception of x_0 and x_1. Such a one is, e.g., $L = [0, 1]$ with the natural ordering of real numbers, and $\ominus$ is the usual difference of real numbers in the interval [0, 1].

In $\mathscr{E}(H)$ there are plenty of regular observables, for example, those having range in $L(H)$. We note that an observable x is regular iff $A \in \mathcal{R}(x)$, $A \neq O$, implies $A, I - A \nleq \frac{1}{2}I$ (equivalently, the spectrum of A and $I - A$ is not contained in [0, 1/2]).

8. ORTHOMODULARITY IN DIFFERENCE POSETS

In the present section, we derive necessary and sufficient conditions that a subset of a D-poset L satisfies orthomodularity, and the given results generalize those in Lahti and Maczyński (1992).

A triplet of mutually orthogonal elements (not necessary different) a_1, a_2, a_3 of a D-poset L is said to be a *triangle*, and is denoted by $\triangle(a_1, a_2, a_3)$. A triangle $\triangle(a_1, a_2, a_3)$ is said to be *closed* if $a_1 \oplus a_2 \oplus a_3$ is an element of L. This is equivalent to saying $\{a_1, a_2, a_3\}$ is $\bigoplus$-orthogonal.

Let A be a subset of a D-poset L which contains at least one triangle. We say that A is *triangle closed*, or $\triangle$-*closed*, for short, if (i) $a^\perp \in A$

whenever $a \in A$, and (ii) if $a_1, a_2, a_3 \in A$, $\triangle(a_1, a_2, a_3)$, then $a_1 \oplus a_2 \oplus a_3 \in A$, $a_i \oplus a_j \in A$ for $1 \le i < j \le 3$.

We recall that if A is $\triangle$-closed, then 0 and 1 belong to A. Indeed, there exists a triangle $\triangle(a_1, a_2, a_3)$ in A. Then $b = a_1 \oplus a_2 \in A$; consequently, $b^\perp \in A$. Observing that $\{a_1, a_2, b^\perp\} \subset A$ is a triangle, we have $1 = a_1 \oplus a_2 \oplus b^\perp \in A$ and $0 = 1^\perp \in A$.

In a subset A of a D-poset L the *orthomodularity* holds if for $a \le b$, $a, b \in A$, we have $b = a \vee_A (b \wedge_A a^\perp)$, where the meet and join $\wedge_A, \vee_A$ are taken in the set A.

Theorem 8.1. A subset A of a D-poset L is an orthomodular poset with respect to 0, 1, $\le$, $\perp$ if and only if A is $\triangle$-closed. Moreover, $a_1 \vee \cdots \vee a_n = a_1 \oplus \cdots \oplus a_n$ for any set $\{a_1, \ldots, a_n\}$ of pairwise orthogonal elements of L.

Proof. The necessity is clear. For the sufficiency we claim to show that $a \vee b$ exists in A whenever $a \perp b$, and $a \vee b = a \oplus b$.

Since $a \oplus b \ge a, b$, we have to show that if $a, b \le c$ for some $c \in A$, then $a \oplus b \le c$. Since $\{a, b, c^\perp\}$ is a triangle, $a \oplus b \oplus c^\perp \le 1$, hence $a \oplus b \le 1 \ominus c^\perp = c$.

Moreover, by induction, we have $a_1 \vee \cdots \vee a_n = a_1 \oplus \cdots \oplus a_n$, and by the de Morgan law, $a \wedge a^\perp = 0$ for any $a \in A$.

Let now $a \le b$; then $a \vee b^\perp = a \oplus b^\perp = (b \ominus a)^\perp$ and $b = a \oplus (b \ominus a) = a \vee (a \vee b^\perp)^\perp$. By the de Morgan law in A, we have $(a \vee b^\perp)^\perp = (b \wedge a^\perp)$, so that $b = a \vee (b \wedge a^\perp)$. ■

Corollary 8.2. Let A_1 and A_2 be two $\triangle$-closed subsets of L. Then for any $a, b \in A_1 \cap A_2$, if $a \perp b$, then $a \vee_1 b = a \oplus b = a \vee_2 b$.

Proof. It follows easily from Theorem 8.1. ■

We say that a D-subposet A of a D-poset L is an *orthomodular poset* of L if (i) $a, b \in A$, $a \perp b$, then $a \vee_A b \in A$, and (ii) orthomodularity holds in A.

A subset A of L is said to be a *suborthoalgebra* of L if (i) $1 \in A$, (ii) $a \in A$ implies $a^\perp \in A$, (iii) $a, b \in A$, $a \perp b$, then $a \oplus b \in A$, and (iv) $a \oplus a$ for $a \in A$ entails $a = 0$. It is easy to see, owing to the end of Section 4, that a D-subposet A of L is a suborthoalgebra of L iff $a \le a^\perp$, $a \in A$, implies $a = 0$.

Theorem 8.3. Let x be an (σ-) observable of a difference poset L. The following assertions are equivalent:

(i) x is regular.
(ii) $\mathcal{R}(x)$ is a Boolean (σ-) algebra of L.

(iii) $\mathscr{R}(x)$ is $\triangle$-closed.
(iv) $\mathscr{R}(x)$ is an orthomodular poset of L.
(v) $\mathscr{R}(x)$ is a suborthoalgebra of L.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has been proved in Theorem 7.4.

(ii) $\Rightarrow$ (iii). $\{x(\emptyset), x(\emptyset), x(\mathbb{R})\}$ is a triangle in $\mathscr{R}(x)$. Let now $\triangle(x(E), x(F), x(G))$ be given. As in the proof of Theorem 7.4, we can choose mutually disjoint Borel subsets E_1, F_1, G_1 so that $x(E) = x(E_1)$, $x(F) = x(F_1)$, $x(G) = x(G_1)$. Then $x(E_1 \cup F_1) = x(E) \oplus x(F)$, $x(E_1 \cup G_1) = x(E) \oplus x(G)$, $x(F_1 \cup G_1) = x(F) \oplus x(G)$, $x(E_1 \cup F_1 \cup G_1) = x(E) \oplus x(F) \oplus x(G) \in \mathscr{R}(x)$, so that $\mathscr{R}(x)$ is $\triangle$-closed.

(iii) $\Rightarrow$ (iv). It follows from Theorem 8.1.

(iv) $\Rightarrow$ (i). Let $x(E) \le x(E^c)$; then for the triangle $\triangle(x(E), x(E^c), x(\emptyset))$ we have $1 = x(\mathbb{R}) = x(E) \oplus x(E^c) \oplus x(\emptyset) = x(E^c)$, hence $x(E) = 0$, which proves the regularity of x.

(v) $\Leftrightarrow$ (i). It now is evident. ∎

9. BOOLEAN POWERS OF DIFFERENCE POSETS

Let L be a nonempty set and B a complete Boolean algebra with 0_B and 1_B as minimal and maximal elements of B. Motivated by Grätzer (1968) and Drossos and Markakis (1994), we say that the structure $L[B]$ defined via

$$L[B] = \left\{ f \in B^L \colon a \neq b \;\Rightarrow\; f(a) \wedge f(b) = 0_B \text{ and } \bigvee_{a \in L} f(a) = 1_B \right\} \tag{9.1}$$

is a *Boolean power* of L (or a *Boolean extension*).

Boolean powers for OMPs and orthoalgebras have been studied in Pulmannová (1993*a*,*b*) and in a modified form in Pták (1986). In the present section we give a Boolean power of a difference poset showing that it is always a D-poset.

Define, for any $a \in L$, a mapping $\hat{a}\colon L \to B$ via

$$\hat{a}(x) = \begin{cases} 1_B & \text{if } x = a, \\ 0_B & \text{if } x \neq a, \end{cases} \qquad x \in L \tag{9.2}$$

Then $\hat{a} \in L[B]$.

Theorem 9.1. Let L be a difference poset with $\le$, 0, 1, and $\ominus$, and B be a Boolean complete algebra. For f, $g \in L[B]$ we define a partial binary relation $\le$ via

$$f \le g \qquad \text{iff} \quad \bigvee_{\substack{x,y \in L \\ x \le y}} f(x) \wedge g(y) = 1_B \tag{9.3}$$

and for $f, g \in L[B]$ with $f \le g$ we put $f \ominus g: L \to B$ via

$$(f \ominus g)(x) = \bigvee_{\substack{a,b \in L \\ x = b \ominus a}} f(b) \wedge g(a), \qquad x \in L \tag{9.4}$$

The Boolean power $L[B]$ with $\le$, $\ominus$ defined via (9.1), (9.3), and (9.4) is a difference poset with the minimal and maximal elements $\hat{0}$ and $\hat{1}$, respectively.

Proof. First of all we show that $\le$ defined by (9.3) is a partial ordering on $L[B]$.

Reflexivity. Since $f(x) \le f(x)$ for any $x \in L$, we have

$$\bigvee_{\substack{x,y \in L \\ x \le y}} f(x) \wedge f(y) = \bigvee_{x \in L} f(x) = 1_B$$

Antisymmetry. Let now $f \le g$, $g \le f$; then

$$\begin{aligned} 1_B &= \left[\bigvee_{\substack{x,y \in L \\ x \le y}} [f(x) \wedge g(y)]\right] \wedge \left[\bigvee_{\substack{u,v \in L \\ u \le v}} [g(u) \wedge f(v)]\right] \\ &= \bigvee_{\substack{x,y \in L \\ x \le y}} \bigvee_{\substack{u,v \in L \\ u \le v}} [f(x) \wedge g(y) \wedge g(u) \wedge f(v)] \\ &= \bigvee_{\substack{x,y,v \in L \\ x \le y \le v}} [f(x) \wedge g(y) \wedge f(v)] = \bigvee_{x \in L} [f(x) \wedge g(x)] \end{aligned}$$

so that, for any $y \in L$,

$$f(y) = \left(\bigvee_{x \in L} f(x) \wedge g(x)\right) \wedge f(y) = \bigvee_{x \in L} [f(x) \wedge f(y) \wedge g(x)] = f(y) \wedge g(y)$$

Hence $f(y) \le g(y)$. By symmetry, $g(y) \le f(y)$; hence $f(y) = g(y)$, $y \in L$, and $f = g$.

Transitivity. Let $f, g, h \in L[B]$ with $f \le g$ and $g \le h$ be given. Calculate

$$\begin{aligned} 1_B &= \left[\bigvee_{\substack{x,y \in L \\ x \le y}} [f(x) \wedge g(y)]\right] \wedge \left[\bigvee_{\substack{u,v \in L \\ u \le v}} [g(u) \wedge h(v)]\right] \\ &= \bigvee_{\substack{x,y \in L \\ x \le y}} \bigvee_{\substack{u,v \in L \\ u \le v}} [f(x) \wedge g(y) \wedge g(u) \wedge h(v)] \\ &= \bigvee_{\substack{x,y,v \in L \\ x \le y \le v}} [f(x) \wedge g(y) \wedge h(v)] \\ &\le \bigvee_{\substack{x,y \in L \\ x \le v}} [f(x) \wedge h(v)] \wedge \left(\bigvee_{y \in L} g(y)\right) \\ &= \bigvee_{\substack{x,v \in L \\ x \le v}} [f(x) \wedge h(v)] \end{aligned}$$

i.e., $f \le h$.

From the above we conclude that $f = g$ iff $\bigvee_{x\in L} f(x) \wedge g(x) = 1_B$.

Let now $f \in L[B]$ and $0, 1 \in L$ be maximal and minimal elements of L. Then

$$\bigvee_{\substack{x,y\in L\\ x\le y}} [\hat{0}(x) \wedge f(y)] = \bigvee_{y\in L} f(y) = 1_B$$

Hence $\hat{0} \le f$. Similarly

$$\bigvee_{\substack{x,y\in L\\ x\le y}} [f(x) \wedge \hat{1}(y)] = \bigvee_{x\in L} f(x) = 1_B$$

so that $f \le \hat{1}$.

In other words, $L[B]$ with $\le$ defined by (9.3) is a poset with the least and greatest elements $\hat{0}$ and $\hat{1}$, respectively.

Now we claim to show that $L[B]$ is a D-poset with $\ominus$ defined by (9.4), Indeed, for that it is necessary to show that $f \ominus \hat{0} = f$ for any $f \in L[B]$, and if $f \le g \le h$, then $h \ominus g \le h \ominus f$ and $(h \ominus f) \ominus (h \ominus g) = g \ominus f$. Calculate

$$\bigvee_{x\in L} [f(x) \wedge (f \ominus \hat{0})] = \bigvee \left[f(x) \wedge \bigvee_{\substack{u,v\in L\\ u\ominus v = x}} f(u) \wedge \hat{0}(v) \right]$$
$$= \bigvee_{x\in L} \left[f(x) \wedge \bigvee_{u\in L} f(u) \right] = \bigvee_{x\in L} f(x) = 1_B$$

so that $f \ominus \hat{0} = f$.

We claim to show that if $f \le g$, then (9.4) defines an element of $L[B]$. Let $a \ne b$. Then

$$(g \ominus f)(a) \wedge (g \ominus f)(b)$$
$$= \left(\bigvee_{\substack{x,y\in L\\ y\ominus x = a}} f(x) \wedge g(y) \right) \wedge \left(\bigvee_{\substack{u,v\in L\\ v\ominus u = b}} f(u) \wedge g(v) \right)$$
$$= \bigvee_{\substack{x,y\in L\\ y\ominus x = a}} \bigvee_{\substack{u,v\in L\\ v\ominus u = b}} [f(x) \wedge g(y) \wedge f(u) \wedge g(v)]$$
$$= \bigvee_{\substack{x,y\in L\\ y\ominus x = a}} \bigvee_{\substack{u,v\in L\\ v\ominus u = b}} f(x) \wedge g(y) = 0_B$$

In addition,

$$\bigvee_{x\in L} (g \ominus f)(x) = \bigvee_{x\in L} \bigvee_{\substack{u,v\in L\\ v\ominus u = x}} g(v) \wedge f(u) = \bigvee_{\substack{u,v\in L\\ u\le v}} g(v) \wedge f(u) = 1_B$$

which says that $g \ominus f \in L[B]$.

Let $f \leq g \leq h$; then

$$\bigvee_{\substack{x,y\in L\\ x\leq y}} (h\ominus f)(y) \wedge (h\ominus g)(x)$$
$$= \bigvee_{\substack{x,y\in L\\ x\leq y}} \bigvee_{\substack{u,v\in L\\ v\ominus u=y}} \bigvee_{\substack{s,t\in L\\ t\ominus s=x}} [h(v)\wedge f(u)\wedge h(t)\wedge g(s)]$$
$$= \bigvee_{\substack{x,y\in L\\ x\leq y}} \bigvee_{\substack{u,s,v\in L\\ v\ominus u=y\\ v\ominus s=x}} [h(v)\wedge g(s)\wedge f(u)]$$
$$= \bigvee_{\substack{u,s,v\in L\\ u\leq s\leq v}} [h(v)\wedge g(s)\wedge f(u)]$$
$$= \left[\bigvee_{\substack{s,v\in L\\ s\leq v}} h(v)\wedge g(s)\right] \wedge \left[\bigvee_{\substack{u,t\in L\\ u\leq t}} g(t)\wedge f(u)\right] = 1_B$$

Hence $h\ominus g \leq h\ominus f$ and

$$\bigvee_{z\in L} [(h\ominus f)\ominus(h\ominus g)(z)\wedge(g\ominus f)(z)]$$
$$= \bigvee_{z\in L} \bigvee_{\substack{x,y\in L\\ y\ominus x=z}} \bigvee_{\substack{p,q\in L\\ q\ominus p=z}} [(h\ominus f)(y)\wedge(h\ominus g)(x)\wedge g(q)\wedge f(p)]$$
$$= \bigvee_{z\in L} \bigvee_{\substack{x,y\in L\\ y\ominus x=z}} \bigvee_{\substack{u,v\in L\\ v\ominus u=y}} \bigvee_{\substack{s,t\in L\\ t\ominus s=x}} \bigvee_{\substack{p,q\in L\\ q\ominus p=z}} [h(v)\wedge f(u)\wedge h(t)\wedge g(s)\wedge g(q)\wedge f(p)]$$
$$= \bigvee_{z\in L} \bigvee_{\substack{x,y\in L\\ y\ominus x=z}} \bigvee_{\substack{u,s,v\in L\\ u\leq s\leq v\\ v\ominus u=y\\ v\ominus s=x\\ s\ominus u=z}} [h(v)\wedge f(u)\wedge g(s)]$$
$$= \bigvee_{z\in L} \bigvee_{\substack{x,y\in L\\ y\ominus x=z}} (h\ominus f)(y)\wedge(h\ominus g)(x)$$
$$= \bigvee_{\substack{x,y\in L\\ x\leq y}} (h\ominus f)(y)\wedge(h\ominus g)(x) = 1_B$$

which proves $(h\ominus f)\ominus(h\ominus g) = g\ominus f$. ■

Let now B be a Boolean algebra and L a D-poset. The set

$$L[B]^* = \left\{f\in B^L\colon a\neq b \Rightarrow f(a)\wedge f(b)=0_B, f(L) \text{ is finite}, \bigvee_{a\in L} f(a) = 1_B\right\}$$

is said to be a *bounded Boolean power* of L. Using the same methods as in the proof of Theorem 9.1, we may prove that $L[B]^*$ with $\hat{0}$, $\hat{1}$, $\leq$, $\ominus$ defined by (9.2)–(9.4) is a difference poset, too. For L a logic or an orthoalgebra, bounded Boolean powers (in an equivalent modified form) are studied in Pták (1986) and Foulis and Pták (1993). We recall that the properties discussed in the following section also hold for bounded Boolean powers.

10. PROPERTIES OF BOOLEAN POWERS

Let B be a complete Boolean algebra and L a difference poset. Let $T = \{t_i : i \in I\}$ be a resolution of 1_B, i.e. $t_i \wedge t_j = 0_B$ if $i \neq j$ and $\bigvee_{i \in I} t_i = 1_B$. If $\{f_i : i \in I\} \subseteq L[B]$, then

$$f(x) = \bigvee_{i \in I} (f_i(x) \wedge t_i), \qquad x \in L \tag{10.1}$$

is an element of $L[B]$. For (10.1) we can use $f = \bigvee_{i \in I} f_i \wedge t_i$ or the "sum" notation

$$f = \sum_{i \in I} f_i \cdot t_i \tag{10.2}$$

In particular, if $\{a_i : i \in I\} \subseteq L$ and $\{t_i : i \in I\}$ is a resolution of 1_B, then

$$\sum_{i \in I} \hat{a}_i \cdot t_i \tag{10.3}$$

belongs to $L[B]$. Conversely, any element $f \in L[B]$ can be written in the form (10.3) for appropriate pairwise different a_i in L and a resolution $T = \{t_i : i \in I\}$. Indeed, given $f \in L[B]$, we put $I = L$ and $T = \{f(a) : a \in L\}$. Then

$$f = \sum_{a \in L} \hat{a} \cdot t_a$$

where $t_a = f(a)$, $a \in L$.

In addition, we may assume that the resolution of 1_B is strictly positive (i.e., $t_i \neq 0_B$ for each i). This form is called the *reduced representation* of f by its values, supposing that the a_i are pairwise different. We recall that in this case f has a unique reduced representation. Indeed, if $f = \sum_i \hat{a}_i \cdot t_i = \sum_j \hat{b}_j \cdot s_j$, $t_i, s_j > 0_B$, and $\{a_i\}$ and $\{b_j\}$ consist of pairwise different elements, then $f(x) = 0_B$ iff $x \neq a_i$ for any i, and $f(x) = t_i$ iff $x = a_i$, so that $a_i = b_j$ and $t_i = s_j$ for some i and j.

Let L_1 and L_2 be two difference posets. The mapping $h: L_1 \to L_2$ is called a *monomorphism* if (i) $h(1_1) = 1_2$, and (ii) $a \perp b$ iff $h(a) \perp h(b)$, and $h(a \oplus b) = h(a) \oplus h(b)$. Then $h(a^\perp) = h(a)^\perp$, $a \in L$, and h is injective. If h is surjective, we say that L_1 is *isomorphic* with L_2 (or, without misunderstanding, that L_1 and L_2 are isomorphic).

Theorem 10.1. Let L be a difference poset and let B be a complete Boolean algebra. Then the mappings $\lambda: L \to L[B]$, defined via $\lambda(a) = \hat{a}$, where $\hat{a}$ is defined via (9.2), and $\beta: B \to L[B]$ defined for $b \in B$ via

$$\beta(b)(x)\begin{cases} b & \text{if} \quad x = 1_L \\ b^c & \text{if} \quad x = 0_L, \quad x \in L \\ 0_B & \text{otherwise} \end{cases} \tag{10.4}$$

are monomorphisms preserving all existing suprema (infima) in L and $L[B]$, respectively. In particular

$$\lambda\left(\bigoplus_i a_i\right) = \bigoplus_i \lambda(a_i)$$

whenever $\bigoplus_i a_i$ exists in L.

Proof. We recall that the partial binary operation $\oplus$ on $L[B]$ can be defined either via (2.1) or, equivalently, via

$$(f \oplus g)(x) = \bigvee_{\substack{u,v \in L \\ u \oplus v = x}} f(u) \wedge g(v), \qquad x \in L$$

and $f^{\perp}(x) = f(x^{\perp})$, $x \in L$.

Since $\lambda(a) \leq \lambda(b)$ iff $a \leq b$, we conclude that if $a = \bigvee_i a_i$, then $\lambda(a) \geq \lambda(a_i)$ for any i. If for $g \in L[B]$ we have $g \geq \lambda(a_i)$ for any i, we have

$$1_B = \bigvee_{\substack{x,y \in L \\ x \leq y}} g(y) \wedge \hat{a}_i(x) = \bigvee_{\substack{y \in L \\ y \geq a_i}} g(y)$$

which gives

$$1_B = \bigvee_{\substack{y \in L \\ y \geq a}} g(y) = \bigvee_{\substack{x,y \in L \\ x \leq y}} g(y) \wedge \hat{a}(x)$$

so that $\lambda(a) \leq g$.

For β we conclude as follows. We have $\beta(b)(x) = (\hat{1}(x) \wedge b) \vee (\hat{0}(x) \wedge b^c)$ or $\beta(b) = \hat{1} \cdot b + \hat{0} \cdot b^c$. Hence $\beta(1)(x) = \hat{1}(x)$ and $\beta(b^c)(x) = (\hat{1}(x) \wedge b^c) \vee (\hat{0}(x) \wedge b) = \beta(b)(x^{\perp})$, since $\hat{1}(x^{\perp}) = \hat{0}(x)$ and $\hat{0}(x^{\perp}) = \hat{1}(x)$. Let $f \in L[B]$. Then

$$\beta(b)(x) \wedge f(y) = \begin{cases} b \wedge f(y) & \text{if} \quad x = 1_L \\ b^c \wedge f(y) & \text{if} \quad x = 0_L \\ 0_B & \text{if} \quad x \neq 1_L, \quad x \neq 0_L \end{cases}$$

and therefore

$$\bigvee_{\substack{x,y \in L \\ x \leq y}} [\beta(b)(x) \wedge f(y)] = (b \wedge f(1)) \vee \left(\bigvee_{y \in L} [b^c \wedge f(y)]\right) = b^c \vee (b \wedge f(1))$$

which entails that $\beta(b) \leq f$ if and only if $b \leq f(1)$. For b_1, $b_2 \in B$ this gives that $\beta(b_1) \leq \beta(b_2)$ if and only if $b_1 \leq b_2$. Now assume that $b = \bigvee_i b_i$. Let f be any upper bound of $\beta(b_i)$ for any i. From $b_i \leq f(1)$ for any i we get $\bigvee_i b_i \leq f(1)$, hence $\beta(b) \leq f$. That is, $\beta(b) = \bigvee_i \beta(b_i)$. This proves that $\beta: B \to L[B]$ is a complete embedding. ■

Theorem 10.2. Let μ be a finitely additive state on L and η a completely additive state on B. Then there is a unique finitely additive state, $\mu \otimes \eta$, on $L[B]$ such that

$$(\mu \otimes \eta)(f) = \sum_i \mu(a_i)\eta(t_i) \tag{10.5}$$

whenever f is in the reduced representation form $f = \sum_i \hat{a}_i \cdot t_i$.

Proof. Since the reduced representation form of f is unique, the right-hand side of (10.5) is defined correctly, and $0 \leq (\mu \otimes \eta)(f) \leq 1$ for any $f \in L[B]$. Due to $\hat{1} = \hat{1} \cdot 1_B$, we conclude that $(\mu \otimes \eta)(\hat{1}) = 1$. In addition, if f has any (no reduced representation $f = \sum_j \hat{b}_j \cdot s_j$, then, for any i we have $0_B \neq f(a_i) = t_i = \bigvee_j \hat{b}_j(a_i) \wedge s_j$, which gives that, for any i, there is j such that $t_i = s_j$ and therefore $(\mu \otimes \eta)(f) = \sum_j \mu(b_j)\eta(s_j)$, too.

Suppose that $f \perp g$, $f, g \in L[B]$; then we can assume that f and g are of the form $f = \sum_i \hat{u}_i \cdot w_i$ and $g = \sum_i \hat{v}_i \cdot w_i$, where $u_i \perp v_i$, and hence $f \oplus g = \sum_i (u_i \otimes v_i)^\wedge \cdot w_i$, and

$$\begin{aligned}(\mu \otimes \eta)(f \oplus g) &= \sum_i \mu(u_i \oplus v_i)\eta(w_i) \\ &= \sum_i \mu(u_i)\eta(w_i) + \sum_i \mu(v_i)\eta(w_i) \\ &= (\mu \otimes \eta)(f) + (\mu \otimes \eta)(g) \quad ■\end{aligned}$$

The state $\mu \otimes \eta$ from Theorem 10.3 is said to be a *product state* of μ and η on $L[B]$.

If $f = \sum_i \hat{a}_i \cdot t_i$, $g = \sum_j \hat{b}_j \cdot s_j$ are reduced representations, then $f \leq g$ iff $a_i \leq b_j$ whenever $t_i \wedge s_j \neq 0_B$. Consequently, $\beta(b) \wedge f$ exists for any $b \in B$ and $f \in L[B]$, and is equal to

$$\beta(b) \wedge f = \sum_i \hat{a}_i \cdot t_i \wedge b + \hat{0} \cdot t_i \wedge b^c$$

Moreover, $\beta(b) \wedge (f \oplus g) = \beta(b) \wedge f \oplus \beta(v) \wedge g$.

For a state m on $L[B]$ ($L[B]^*$) define the mapping $m_{\beta(b)}$ via

$$m_{\beta(b)}(f) = m(\beta(b) \wedge f), \qquad f \in L[B] \quad (f \in L[B]^*)$$

11. EXAMPLES OF BOOLEAN POWERS

In the present section, we give three special cases of Boolean powers when (i) B is an atomic complete Boolean algebra, (ii) L is the set of all effects of a von Neumann algebra, and (iii) B is a set of skew operators.

We recall that a Boolean algebra B is *atomic* if, for any $a \in B$, there is an *atom* b of B such that $b \le a$. Denote by B_o the set of all atoms in B. Then B_o is the resolution of 1_B.

Let, for any $i \in I$, L_i with $\le_i$, 1_i, $\ominus_i$ be a D-poset. Then $L := \prod_{i \in I} L_i$ is a D-poset, called a *product D-poset* of $\{L_i : i \in I\}$, when $\le$, 1, and $\ominus$ are defined on L as follows: $\{a_i\} \le \{b_i\}$ iff $a_i \le_i b_i$, $i \in I$, $1 = \{1_i\}$, $\{b_i\} \ominus \{a_i\} = \{b_i \ominus_i a_i\}$.

Motivated by Grätzer (1968), we can prove the following statement:

Theorem 11.1. Let B be an atomic, complete Boolean algebra and L a D-poset. Let $L_B = \prod_{b \in B_o} L_b$, where $L_b = L$ for every $b \in B_o$. Then L_B is isomorphic with $L[B]$.

Proof. For $\{a_b\} \in L_B$ we define an element of $L[B]$ via $\sum_{b \in B_o} \hat{a}_b \cdot b$. It is not hard to show that the mapping $h: L_B \to L[B]$ such that $\{a_b\} \mapsto \sum_{b \in B_o} \hat{a}_b \cdot b$ is a monomorphism from L_B into $L[B]$.

On the other hand, let $f \in L[B]$ and let it have the reduced representation $f = \sum_i \hat{a}_i \cdot t_i$. Then, for any $x \in L$, we have

$$
\begin{aligned}
f(x) &= \bigvee_i \hat{a}_i(x) \wedge t_i \\
&= \bigvee_i \bigvee_{b \in B_o} \hat{a}_i(x) \wedge t_i \wedge b \\
&= \bigvee_{b \in B_o} \bigvee_{\substack{i \\ b \le t_i}} \hat{a}_i(x) \wedge t_i \wedge b \wedge \bigvee_{b \in B_o} \bigvee_{\substack{i \\ b \perp t_i}} \hat{a}_i(x) \wedge t_i \wedge b \\
&= \bigvee_{b \in B} \bigvee_{\substack{i \\ b \le t_i}} \hat{a}_i(x) \wedge t_i \wedge b
\end{aligned}
$$

If we put $a_b = a_i$ whenever $b \le t_i$, then $f = \sum_{b \in B_o} \hat{a}_b \cdot b$, which proves that h is surjective. ■

Corollary 11.2. Let μ be a σ-additive or completely additive state on a D-poset L and η a completely additive state on an atomic, complete Boolean algebra B. Then the product state $\mu \otimes \eta$ on $L[B]$ is σ-additive or completely additive, respectively.

Proof. The assertion follows from Theorem 11.1 because then $f = \bigotimes_i f_i$ iff $f_i = \sum_{b \in B_o} \hat{a}_b^i \cdot b$, $f = \sum_{b \in B_o} \hat{a}_b \cdot b$, and $a_b = \bigoplus_i a_b^i$. Owing to (10.5), we obtain the desired result. ■

Suppose that $\mathscr{A}$ is a von Neumann algebra of operators acting on a complex Hilbert space H. Denote by $\mathscr{E}(\mathscr{A})$ the set of all Hermitian operators A on H from $\mathscr{A}$ such that $O \le A \le I$. Then $\mathscr{E}(\mathscr{A})$ with the usual $\le$ and $\ominus$ (as a difference) is a D-poset [in fact, it is a D-subposet of $\mathscr{E}(H)$].

If B is a complete Boolean subalgebra of the set $P(H)$ of all orthogonal projections on H, then by Bade's theorem (Bade, 1955; Dunford and Schwartz, 1971, Section XVII.3), B is the projection lattice of the Abelian von Neumann algebra $\mathscr{A}$ of operators acting on H generated by B, i.e., $B = P(\mathscr{A}) = \{P \in \mathscr{A}: P^* = P, P^2 = P\}$.

Theorem 11.3. Let $L = \mathscr{E}(\mathscr{A}_1)$ be the set of all effects of a von Neumann algebra $\mathscr{A}_1$ of operators acting on a complex Hilbert space H_1 and let $B = P(\mathscr{A}_2)$ be the projection lattice of an Abelian von Neumann algebra $\mathscr{A}_2$ of operators acting on a Hilbert space H_2. Then $L[B]^*$ and $L[B]$ are isomorphic with D-subposets $L_o := \{\sum_{i=1}^n P_i \otimes B_i: P_i \in \mathscr{E}(\mathscr{A}_1)$, $\{B_i\}_{i=1}^n$ is a finite resolution from B of the identity $I_2\}$ of $\mathscr{E}(H_1 \otimes H_2)$ and $L_1 = \{\sum_i P_i \otimes B_i: P_i \in \mathscr{E}(\mathscr{A}_1)$, $\{B_i\}$ is any resolution from B of $I_2\}$, respectively.

Proof. (i) Let us consider the set L_o of all operators on $H_1 \otimes H_2$ of the form $\sum_{i=1}^n P_i \otimes B_i$, where $P_i \in \mathscr{E}(\mathscr{A}_1)$ and $\{B_i\}_{i=1}^n$ is a finite resolution of the identity I_2. Then L_o is a D-subposet of $\mathscr{E}(\mathscr{A}_1 \otimes \mathscr{A}_2) \subseteq \mathscr{E}(H_1 \otimes H_2)$. Indeed, the minimal element of L_o is of the form $O_1 \otimes B$ or $P \otimes O_2$, where O_i is the zero operator on H_i and $P \in \mathscr{E}(\mathscr{A}_1)$, $B \in P(\mathscr{A}_2)$. We have to verify the condition of Remark 2.2.

Let $P = \sum_{i=1}^n P_i \otimes B_i$; then

$$P \ominus O = \sum_{i=1}^{n} P_i \otimes B_i \ominus O_1 \otimes I_2 = \sum_{i=1}^{m} (P_i \ominus O_1) \otimes B_i = \sum_{i=1}^{n} P_i \otimes B_i = P$$

We claim that for $P, Q \in L_o$, $P = \sum_{i=1}^n P_i \otimes B_i$, $Q = \sum_{j=1}^m Q_j \otimes C_j$ we have $P \leq Q$ iff $P_i \leq Q_j$ whenever $B_i C_j \neq O_2$. Indeed, let $P \leq Q$; then $O \leq Q \ominus P = \sum_{i,j} (Q_j \ominus P_i) \otimes B_i C_j$. Let $\phi \in H_1$, $0 \neq \psi \in B_{i_o} C_{j_o}$. Then

$$O \leq \langle (Q \ominus P)\phi \otimes \psi, \phi \otimes \psi \rangle = \langle (Q_{j_o} \ominus P_{i_o})\phi, \phi \rangle \|\psi\|^2$$

which entails $P_{i_o} \leq Q_{j_o}$. Conversely, if $P_{i_o} \leq Q_{j_o}$ whenever $B_{i_o} C_{j_o} \neq O_2$, then easily $P \leq Q$.

In addition, if all P_i are mutually different, then the representation of P in the form $P = \sum_{i=1}^n P_i \otimes B_i$ $(B_i \neq O_2)$ is unique. Indeed, assume $P = \sum_{i=1}^n P_i \otimes B_i = \sum_{j=1}^m Q_j \otimes C_j$. We have proved that $P_i = Q_j$ whenever $B_i C_j \neq O_2$. Suppose now that $B_{i_o} C_{j_o} \neq O_2$ and choose $\psi \in C_{j_o}$. Then, for any $\phi \in H_1$,

$$\begin{aligned} P_{i_o} \otimes B_{i_o} \phi \otimes \psi = P\phi \otimes \psi &= \left(\sum_{i=1}^{n} P_i \otimes B_i \right) \phi \otimes \psi \\ &= \left(\sum_{j=1}^{m} Q_j \otimes C_j \right) \phi \otimes \psi = Q_{j_o} \otimes C_{j_o} \phi \otimes \psi \end{aligned}$$

which gives

$$P_{i_o}\phi \otimes B_{i_o}\psi = Q_{j_o}\phi \otimes C_{j_o}\psi = Q_{j_o}\phi \otimes \psi$$

and consequently $C_{j_o} \le B_{i_o}$ and by symmetry $B_{i_o} \le C_{j_o}$.

Therefore, (ii) of Remark 2.2 easily holds.

(ii) For this we recall that the series $\sum_i P_i \otimes B_i$, where $\{B_i\}$ is any resolution of I_2, strongly converges, and it is an element of $\mathscr{E}(\mathscr{A}_1 \otimes \mathscr{A}_2)$. Using similar arguments to those in the first part of the present proof, we conclude that L_1 is a D-subposet of $\mathscr{E}(\mathscr{A}_1 \otimes \mathscr{A}_2)$.

The mapping h from $L[B]^*$ or $L[B]$ into L_o or L_1, respectively, defined via $h(\sum_i \hat{P}_i \cdot B_i) = \sum_i P_i \otimes B_i$ is the desired monomorphism. ■

Let now P be a skew operator on a complex Hilbert space H, i.e., P is a bounded linear operator on H with $P^2 = P$. Denote by $L_{\mathscr{S}}(H)$ the set of all skew operators on H. For two skew operators P and Q on H we write $P \le Q$ iff $PQ = QP = P$ (which is equivalent to the requirement $M_P \subseteq M_Q$ and $N_Q \subseteq N_P$, where $M_P = \{Px : x \in H\}$ $(= \{x \in H: Px = x\})$ and $N_P = \{x \in H: Px = 0\}$.[5] Then $O \le P \le I$ for any $P \in L_{\mathscr{S}}(H)$ and for any P we define an orthocomplement $P^\perp$ of P via $P^\perp = I - P$, which entails that (Mushtari, 1989; Mushtari and Matvejchuk, 1985) $L_{\mathscr{S}}(H)$ with respect to $\le$, $\perp$, O, and I is an OMP which, if $\dim H \ge 3$ (Mushtari, 1989), is not a lattice, and if $\dim H = \infty$, then it is not a σ-OMP, respectively. We recall that $P \perp Q$ iff $P + Q \le I$ (equivalently, $QP = PQ = O$), and then $P \vee Q = P + Q$ and $P \wedge Q = PQ$, and $M_{P \wedge Q} = M_P \cap M_Q$ when $PQ = QP$. Let B be a complete Boolean algebra of skew operators. Then by Bade's result (Dunford and Schwartz, 1971, pp. 2196–2199), B is the set of all skew operators of the Abelian von Neumann algebra $\mathscr{A}$ generated by B, and, in addition, there is a constant $K \ge 0$ such that $\|P\| \le K$ for any $P \in B$.

We denote by $L_{\mathscr{S}}(\mathscr{A})$ the set of all skew operators on H from $\mathscr{A}$. The $\mathscr{L}_{\mathscr{S}}(\mathscr{A})$ is a sub-OMP of $L_{\mathscr{S}}(H)$.[6]

Theorem 11.4. Let $L_{\mathscr{S}}(\mathscr{A}_1)$ be the set of all skew operators of a von Neumann algebra $\mathscr{A}_1$ of operators acting on a complex Hilbert space H_1, and let B be a Boolean algebra of skew operators on a complex Hilbert space H_2. Then $L[B]^*$ is isomorphic with the sub *OMP* $L_o = \{\sum_{i=1}^n P_i \otimes B_i : P_i \in L_{\mathscr{S}}(\mathscr{A}_1),\ \{B_i\}_{i=1}^n$ is a resolution from B of $I_2\}$ of the OMP $L_{\mathscr{S}}(H_1 \otimes H_2)$.

[5] P projects H onto M_P parallel to N_Q.

[6] A subset L_o of an OMP L is a *sub* OMP of L if (i) $0, 1 \in L_o$; (ii) $a^\perp \in L_o$ if $a \in L_o$; (iii) $a \vee b \in L_o$ ($\vee$ taken in L) if $a \perp b$, $a, b \in L_o$.

Proof. Let $\phi \in H_1$, $\psi \in H_2$; then

$$P^2\phi \otimes \psi = \sum_{i,j} P_iP_j\phi \otimes B_iB_j\psi = \sum_i P_i^2\phi \otimes B_i^2\psi = \sum_i P_i\phi \otimes B_i\psi = P\phi \otimes \psi$$

so that $P^2 = P$.

Since $I = I_1 \otimes I_2 = \sum_{i=1}^n I_1 \otimes B_i$, we conclude that $I - P = \sum_{i=1}^n (I_1 - P_i) \otimes B_i \in L_o$.

We claim that if $P, Q \in L_o$, $P = \sum_{i=1}^n P_i \otimes B_i$ and $Q = \sum_{j=1}^m Q_j \otimes C_j$, we have $P \leq Q$ iff $P_i \leq Q_j$ whenever $B_iC_j \neq O_2$. Indeed, let $P \leq Q$ and $B_{i_o}C_{j_o} \neq O_2$. Then $B_{i_o}C_{j_o} = B_{i_o} \wedge C_{j_o}$, and

$$M_{P_{i_o} \wedge Q_{j_o}} = M_{B_{i_o}} \cap M_{C_{j_o}}$$

For

$$\phi \in M_{P_{i_o}}, \qquad \psi \in M_{B_{i_o}} \cap M_{C_{j_o}}, \qquad \psi \neq 0$$

we have $P\phi \otimes \psi = \phi \otimes \psi$, so that $Q\phi \otimes \psi = \phi \otimes \psi$ and

$$\begin{aligned} 0 &= \|(Q - P)\phi \otimes \psi\|^2 = \|(Q_{j_o} - P_{i_o}) \otimes B_{i_o}C_{j_o}\phi \otimes \psi\|^2 \\ &= \|Q_{j_o}\phi - P_{i_o}\phi\|^2\|\psi\|^2 = \|Q_{j_o}\phi - \phi\|^2\|\psi\|^2 \end{aligned}$$

which entails

$$\phi \in M_{Q_{j_o}}.$$

Similarly, $Q^\perp \leq P^\perp$ entails

$$N_{Q_{j_o}} = M_{Q_{j_o}^\perp} \subseteq M_{P_{i_o}^\perp} = N_{P_{i_o}}$$

Consequently $P_{i_o} \leq Q_{j_o}$.

Conversely, if $P_i \leq Q_j$ whenever $B_iC_j \neq O_2$, then $PQ = \sum_{i,j} P_iQ_j \otimes B_iQ_j = \sum_{i,j} P_i \otimes B_iC_j = P$, and similarly $QP = P$.

In addition, if P_i are mutually different, then $P = \sum_{i=1}^n P_i \otimes B_i$ has a unique representation in this form. Indeed, let $P = \sum_{i=1}^n P_i \otimes B_i = \sum_{j=1}^m Q_j \otimes C_j$. Then $P_i = Q_j$ if $B_iC_j \neq O_2$. Suppose now that $B_{i_o}C_{j_o} \neq O_2$ and choose

$$\psi \in M_{C_{j_o}}$$

Then, for any

$$\phi \in M_{P_{i_o}}$$

we have

$$P\phi \otimes \psi = \sum_j Q_j\phi \otimes C_j\psi = Q_{j_o}\phi \otimes C_{j_o}\psi = P_{i_o}\phi \otimes \psi = \phi \otimes \psi$$

and

$$P\phi\otimes\psi=\left(\sum_i P_i\otimes B_i\right)\phi\otimes\psi=\sum_{i,j}P_iQ_j\phi\otimes B_iC_j\psi$$

$$=\sum_i P_iQ_{j_o}\phi\otimes B_iC_{j_o}\psi=P_{i_o}Q_{j_o}\phi\otimes B_{i_o}C_{j_o}\psi$$

$$=P_{i_o}\phi\otimes B_{ij}\psi=\phi\otimes B_{i_o}\psi$$

which gives $\psi=B_{i_o}\psi$, i.e.,

$$\psi\in M_{C_{j_o}}$$

By symmetry,

$$M_{C_{j_o}}\subseteq M_{B_{i_o}}$$

so that $C_{i_o}=B_{j_o}$.

Therefore, if $P\perp Q$, there are representations of P and Q of the forms $P=\sum_{i=1}^n P_i\otimes B_i$ and $Q=\sum_{i=1}^n Q_i\otimes B_i$, $P_i\perp Q_i$ (for which we use refinements of resolutions of I_2 in B, if necessary), so that

$$P+Q=\sum_{i=1}^n(P_i+Q_i)\otimes B_i=\sum_{i=1}^n(P_i\vee Q_i)\otimes B_i=P\vee Q$$

Finally, the mapping $H\colon\sum_{i=1}^n\hat{P}_i\cdot B_i\mapsto\sum_{i=1}^n P_i\otimes B_i$ is the monomorphism in question from $L[B]^*$ onto L_o. ■

12. D-POSETS AND QUANTUM MEASUREMENTS

According to Busch *et al.* (1991), in the traditional Hilbert space approach, a measurement of an observable x (a POV-measure, in general) of a physical system $\mathcal{S}$ described by a Hilbert space $H_{\mathcal{S}}$ is a quintuplet $\mathcal{M}=(H_{\mathcal{A}},x_{\mathcal{A}},T_{\mathcal{A}},f,V)$, where $H_{\mathcal{A}}$ is a Hilbert space of a measuring apparatus $\mathcal{A}$, $x_{\mathcal{A}}$ is a *pointer observable* (a POV-measure on $H_{\mathcal{A}}$), $T_{\mathcal{A}}$ is an initial state of $\mathcal{A}$, f is a *pointer function*, that is, a measurable function $f\colon\Omega\to\Omega_{\mathcal{A}}$ which correlates the value spaces $(\Omega,\mathcal{F})$ and $(\Omega_{\mathcal{A}},\mathcal{F}_{\mathcal{A}})$ of x and $x_{\mathcal{A}}$, and $V\colon T(H_{\mathcal{S}}\otimes H_{\mathcal{A}})$ is a trace preserving positive linear transformation of the trace-class operators $T(H_{\mathcal{S}}\otimes H_{\mathcal{A}})$ of the composite system $\mathcal{S}+\mathcal{A}$, satisfying the following two requirements:
The first one is as follows:

$$\mathrm{tr}(Tx(F))=\mathrm{tr}[\mathcal{R}_{\mathcal{A}}(V(T\otimes T_{\mathcal{A}}))(x_{\mathcal{A}}(f^{-1}(F)))]\tag{12.1}$$

for any value set $F\in\mathcal{F}$ and for all possible initial states $T\in T(H_{\mathcal{S}})_1^+$. Here $\mathcal{R}_{\mathcal{A}}(V(T\otimes T_{\mathcal{A}}))$ denotes the reduction of the final state of $\mathcal{S}+\mathcal{A}$ to $\mathcal{A}$ via relative trace. A quintuplet $\mathcal{M}$ satisfying (12.1) is called a *premeasurement*.

The second basic requirement for the quintuplet $(H_{\mathscr{A}}, x_{\mathscr{A}}, T_{\mathscr{A}}, f, V)$ to qualify as an x measurement is the objectification requirement: the measurement should lead to a definite result. This condition is satisfied if $x_{\mathscr{A}}$ is a classical observable, i.e., a PV-measure[7] which commutes with all other observables of $\mathscr{A}$.

If also the equation

$$\mathrm{tr}(Tx(F)) = \mathrm{tr}(\mathscr{R}_{\mathscr{S}}(V(T \otimes T_{\mathscr{A}}))(x(F))$$

is satisfied for all $F \in \mathscr{F}$ and all $T \in T(H_{\mathscr{S}})_1^+$, the measurement $\mathscr{M}$ is called a *first-kind measurement*.

All the features of a measurement $\mathscr{M}$ that pertain to the object system $\mathscr{S}$ are summarized in the instrument $I_{\mathscr{M}}$ of the measurement $\mathscr{M}$. The *instrument* $I_{\mathscr{M}}$ is defined as an operation-valued measure $I_{\mathscr{M}}: \mathscr{F} \to \mathscr{L}(T(H_{\mathscr{S}}))^+$ [where $\mathscr{L}(T(H_{\mathscr{S}}))^+$ is the set of all operations, i.e., positive linear transformations of $T(H_{\mathscr{S}})$], defined by

$$I_{\mathscr{M}}(F)T = \mathscr{R}_{\mathscr{S}}(V(T \otimes T_{\mathscr{A}})) \cdot I \otimes x_{\mathscr{A}}(f^{-1}(F))$$

for all $F \in \mathscr{F}$, $T \in T(H_{\mathscr{S}})$. Here $\mathscr{R}_{\mathscr{S}}(V(T \otimes T_{\mathscr{A}}))$ means the reduction of the final state $V(T \otimes T_{\mathscr{A}})$ of $\mathscr{S} + \mathscr{A}$ to the subsystem $\mathscr{S}$. The instrument reproduces the observable x via the equations

$$\mathrm{tr}(Tx(F)) = \mathrm{tr}(I_{\mathscr{M}}(F)T)$$

for all $F \in \mathscr{F}$, $T \in T(H_{\mathscr{S}})_1^+$. Further, it gives the nonnormalized final states $I_{\mathscr{M}}(F)T$ of $\mathscr{S}$ on the condition that the measurement leads to a result in F.

Two measurements are called *equivalent* if the corresponding instruments are equal.

We note that the notion of an instrument can be defined independently of measurement as an operation-valued measure with some characteristic properties (Davies, 1976; Davies and Lewis, 1970; Luczak, n.d.).

A measurement $\mathscr{M}$ (or the corresponding instrument $I_{\mathscr{M}}$) is *repeatable* if

$$\mathrm{tr}(I_{\mathscr{M}}(E)(I_{\mathscr{M}}(F)T)) = \mathrm{tr}(I_{\mathscr{M}}(E \cap F)T)$$

for all $E, F \in \mathscr{F}$ and all $T \in T(H_{\mathscr{S}})_1^+$.

Following the ideas in Pulmannová (1993*a*,*b*, 1994), we will reformulate basic definitions of measurements in the form of Boolean powers of D-posets. We will assume that a quantum system $\mathscr{S}$ is described by a D-σ-poset and a measuring apparatus $\mathscr{A}$ is described by a Boolean algebra B. To describe the coupled system $\mathscr{S} + \mathscr{A}$, we may choose either the Boolean power $L[B]$ (if B is complete) or the bounded Boolean power

[7]That is, a POV-measure whose range consists only of orthogonal projections.

$L[B]^*$. At this stage, it is difficult to see which of these approaches is more appropriate. $L[B]^*$ is simpler, but $L[B]$ in the case of discrete measurement (i.e., if B is atomic) is a D-σ-poset, while in general, $L[B]$ and $L[B]^*$ are only D-posets. In both approaches, the basic definitions of a measurement are similar.

As a physical state space $\mathscr{P}$ of $\mathscr{S} + \mathscr{A}$ we will consider the convex hull of the set of all product states, that is, the set of all elements of the form $\sum_i \alpha_i m_i \otimes \mu_i$, where α_i are positive numbers with sum $1, m \in \mathscr{P}_L$, $\mu \in \mathscr{P}_B$, where $\mathscr{P}_L$ is a convex set of σ-additive states on L and $\mathscr{P}_B$ is a convex set of σ-additive (completely additive) states on B is $\mathscr{S} + \mathscr{A}$ is described by $L[B]^*$ ($L[B]$). Assume that we want to measure an observable x on $\mathscr{S}$. Let $(\Omega, \mathscr{F})$ be the value space of x. We choose a measuring apparatus $\mathscr{A}$ described by B and a σ-observable $x_{\mathscr{A}}$ on B (a so-called *pointer observable*). If the value space $(\Omega_{\mathscr{A}}, \mathscr{F}_{\mathscr{A}})$ of $x_{\mathscr{A}}$ is different from $(\Omega, \mathscr{F})$, choose a measurable function $f: \Omega \to \Omega_{\mathscr{A}}$ (a *pointer function*). If the initial state of $\mathscr{S}$ is m and we choose $m_{\mathscr{A}}$ as the initial state of $\mathscr{A}$, then the initial state of $\mathscr{S} + \mathscr{A}$ will be the product state $m \otimes m_{\mathscr{A}}$. The measurement means an interaction between $\mathscr{S}$ and $\mathscr{A}$, which results in a change of the state of the coupled system. This change will be described by a convexity-preserving transformation $V: \mathscr{P} \to \mathscr{P}$. If $V(m \otimes m_{\mathscr{A}})$ is the final state of $\mathscr{S} + \mathscr{A}$ after the measurement, then the restrictions $V(m \otimes m_{\mathscr{A}}) \circ \lambda$ and $m \otimes m_{\mathscr{A}} \circ \beta$ uniquely describe the final states of $\mathscr{S}$ and $\mathscr{A}$, respectively, where λ and β are defined in Theorem 10.1. A quintuplet $\mathscr{M} = (B, x_{\mathscr{A}}, m_{\mathscr{A}}, f, V)$ will be classified as a *measurement* of x if

$$m(x(F)) = V(m \otimes m_{\mathscr{A}}) \circ \beta(x_{\mathscr{A}}(f^{-1}(F)))$$

for all $F \in \mathscr{F}$ and all initial states m of $\mathscr{S}$. We note that since B is a Boolean algebra, there are no problems with the objectification.

If also the equality

$$m(x(F)) = V(m \otimes m_{\mathscr{A}}) \circ \lambda(x(F))$$

is satisfied for all $F \in \mathscr{F}$ and all initial states m of $\mathscr{S}$, the measurement $\mathscr{M}$ is called a *measurement of the first kind.*

The transformation V can be extended by homogeneity to the positive cone $K(\mathscr{P}) = \{\alpha s: s \in \mathscr{P}, \alpha \geq 0\}$, and then by linearity to $K(\mathscr{P}) - K(\mathscr{P})$.

For every measurement $\mathscr{M}$, an instrument is defined by

$$I_{\mathscr{M}}(F)(m) = V(m \otimes m_{\mathscr{A}})_{\beta(x_{\mathscr{A}}(f^{-1}(F)))} \circ \lambda$$

for all $F \in \mathscr{F}$ and all states $m \in \mathscr{P}_L$. It is easy to see that $I_{\mathscr{M}}(F)(m)$ is a σ-additive measure on L and the state obtained after normalization can be interpreted as the final state of $\mathscr{S}$ after measurement under the condition that the measurement leads to a result in the set F. The instrument reproduces the observable x via the equalities

$$\begin{aligned} I_{\mathcal{M}}(F)(m)(1_L) &= V(m \otimes m_A)_{\beta(x_{\mathcal{A}}(f^{-1}(F)))} \circ \lambda(1_L) \\ &= V(m \otimes m_{\mathcal{A}})(\beta(x_{\mathcal{A}} f^{-1}(F)) \wedge \lambda(1_L)) \\ &= m(x(F)) \end{aligned}$$

for all $F \in \mathcal{F}$ and all $m \in \mathcal{P}_L$.

A measurement $\mathcal{M}$ is called *repeatable* if for all $E, F \in \mathcal{F}$ and all m,

$$I_{\mathcal{M}}(E)(I_{\mathcal{M}}(F)(m))(1_L) = I_{\mathcal{M}}(E \cap F)(m)(1_L)$$

Two measurements $\mathcal{M}_1$ and $\mathcal{M}_2$ are *equivalent* if the corresponding instruments are equal, i.e.,

$$I_{\mathcal{M}_1}(F)(m) = I_{\mathcal{M}_2}(F)(m)$$

for all $F \in \mathcal{F}$ and all m.

The basic result is the following one:

Theorem 12.1. Let a physical system $\mathcal{S}$ be described by a D-σ-poset L and let $\mathcal{A}$ be a measuring apparatus.

(i) If $\mathcal{S} + \mathcal{A}$ is described by $L[B]^*$ for a Boolean σ-algebra B, then for every regular σ-observable x on L, there exists a measurement.

(ii) If $\mathcal{S} + \mathcal{A}$ is described by $L[B]$ for some complete Boolean algebra B, then for every regular σ-observable x whose range $\mathcal{R}(x)$ satisfies the c.c.c. condition [i.e., every chain[8] in $\mathcal{R}(x)$ is at most countable] there exists a measurement.

Proof. Let $(\Omega, \mathcal{F})$ be the value space of x. Let $\mathcal{N} = \{E \in \mathcal{F} : x(E) = 0\}$ be the σ-ideal of x-null sets in $\mathcal{F}$ and let $B = \mathcal{F}/\mathcal{N}$. Now B is isomorphic with $\mathcal{R}(x)$, and hence if $\mathcal{R}(x)$ satisfies c.c.c., B is a complete Boolean algebra, and in addition every σ-additive state on B is completely additive. Let $E \mapsto [E]$ be the canonical homomorphism which assigns to every $E \in \mathcal{F}$ the corresponding equivalence class $[E]$ in B. Define the observable $x_{\mathcal{A}} : F \to B$ by $x_{\mathcal{A}}(E) = [E]$, and let the pointer function f be the identity on Ω. Let $(E_i)_i$ be any countable partition of Ω consisting of elements of $\mathcal{F}$. Let $(m_i)_i$ be any sequence of states on L. Let $m_{\mathcal{A}}$ be an arbitrarily chosen state on B, and define a mapping V by

$$V(m \otimes m_{\mathcal{A}}) = \sum_i m(x(E_i)) m_i \otimes \mu_i$$

[8]A subset M of L is a *chain* if for any pair of a, b of M we have either $a < b$ or $b < a$.

where μ_i is the state on B defined by

$$\mu_i([E]) = \frac{m(x(E \cap E_i))}{m(x(E_i))}$$

for every i. Now extend V by convexity to whole $\mathscr{P}$.

It is easy to see that the quintuplet $(B, x_{\mathscr{A}}, m_{\mathscr{A}}, f, V)$ is a measurement of x. ■

Let us recall some basic properties of an instrument.

Let $\mathscr{M} = (B, x_{\mathscr{A}}, m_{\mathscr{A}}, f, V)$ be a measurement of an observable x with value space $(\Omega, \mathscr{F})$ and let $I_{\mathscr{M}}$ be the corresponding instrument. For every $m \in \mathscr{P}_L$,

$$I_{\mathscr{M}}(\Omega)(m) = V(m \otimes m_{\mathscr{A}}) \circ \lambda$$

and for any $G \in \mathscr{F}$ and any $m \in \mathscr{P}_L$,

$$I_{\mathscr{M}}(G)(m) = V(m \otimes m_{\mathscr{A}})_{\beta(b_G)} \circ \lambda$$

where we put $b_G = x_{\mathscr{A}}(f^{-1}(G))$. Since $V(m \otimes m_{\mathscr{A}})$ is a convex combination of product states, the mapping $I_{\mathscr{M}}(\cdot)(m): \mathscr{F} \to \mathscr{P}_L$ is σ-additive for every fixed $m \in \mathscr{P}_L$, that is,

$$I_{\mathscr{M}}\left(\bigcup_{i \in I} F_i\right)(m) = \sum_{i \in I} I_{\mathscr{M}}(F_i)(m)$$

for every sequence $(F_i)_{i \in I}$ of disjoint sets from $\mathscr{F}$. Similarly as in Pulmannová (1994), the first-kind condition can be rewritten in the form

$$m(x(E)) = I_{\mathscr{M}}(\Omega)(m)(x(E))$$

which is equivalent to

$$I_{\mathscr{M}}(E)(I_{\mathscr{M}}(E^c)(m))(1) = I_{\mathscr{M}}(E^c)(I_{\mathscr{M}}(E)(m))(1) \tag{12.2}$$

for any $m \in \mathscr{P}_L$ and all $E \in \mathscr{F}$ [the proof is the same as in Pulmannová (1994)].

Now let us consider the repeatability condition

$$I_{\mathscr{M}}(E)(I_{\mathscr{M}}(F)(m))(1) = I_{\mathscr{M}}(E \cap F)(m)(1)$$

for all $m \in \mathscr{P}_L$ and all $E, F \in \mathscr{F}$.

This condition is equivalent to

$$I_{\mathscr{M}}(E^c)(I_{\mathscr{M}}(E)(m))(1) = 0 \tag{12.3}$$

for any m and all E. Comparing (12.2) and (12.3), we obtain the following result.

Theorem 12.2. A repeatable measurement is of the first kind.

13. CONCLUDING REMARKS

A Boolean power is a special kind of tensor product of two quantum structures (orthomodular posets, orthoalgebras, D-posets) if one of them happens to be a Boolean algebra. The problems with the existence of an appropriate tensor product of orthomodular posets (or orthomodular lattices) has led to the introduction of orthoalgebras, where a tensor product exists in the case that unital sets of states exist (Foulis and Bennett, n.d.). D-posets are even more general structures than orthoalgebras. Their advantage in comparison with orthoalgebras is that they include the event structure on a Hilbert space, described by the set of effects, and reflect the "fuzzy approach" to quantum mechanics. It is an open question whether a tensor product of D-posets exists.

The formulation of some basic properties of quantum measurements given in Section 12 is an attempt at a more general theory of measurements, where the measuring apparatus is described as a classical object. In contrast with previous attempts (Pulmannová, 1993*a*,*b*, 1994), unsharp observables also are included.

ACKNOWLEDGEMENT

This research is supported by grant G-368 of the Slovak Academy of Sciences, Slovakia.

REFERENCES

Alfsen, E. M. (1972). *Compact Convex Sets and Boundary Integrals*, Springer-Verlag, Berlin.
Bade, W. G. (1955). *Transactions of the American Mathematical Society*, **80**, 345–360.
Birkhoff, G., and von Neumann, J. (1936). *Annals of Mathematics*, **37**, 823–843.
Busch, P., Lahti, P. J., and Mittelstaed, P. (1991). *The Quantum Theory of Measurement*, Springer-Verlag, Berlin.
Davies, E. B., (1976). *Quantum Theory of Open Systems*, Academic Press, New York.
Davies, E. B., and Lewis, J. T. (1970). *Communications in Mathematical Physics*, **17**, 239–260.
Cattaneo, G., and Nisticó, G. (1989). *Fuzzy Sets and Systems*, **33**, 165–190.
Drossos, C.A., and Markakis, G. (1994). *Mathematica Slovaca*, **43**, 1–19.
Dunford, N., and Schwartz, J. T. (1971). *Linear Operators, Part III*, Wiley-Interscience, New York.
Dvurečenskij, A. (1993). *Gleason's Theorem and Its Applications*, Kluwer, Dordrecht, Ister Science Press, Bratislava.
Dvurečenskij, A., and Riečan, B. (1994). *International Journal of Theoretical Physics*, to appear.
Foulis, D. J., and Bennett, M. K. (n.d.). Tensor products of orthoalgebras, *Order*, to appear.
Foulis, D., and Pták, P. (1993). On the tensor product of a Boolean algebra and an orthoalgebra, Preprint.
Foulis, D., Greechie, R. J., and Rüttimann, G. T. (1992). *International Journal of Theoretical Physics*, **31**, 787–807.

Grätzer, G. (1968). *Universal Algebras*, van Nostrand, Princeton, New Jersey.
Holevo, A. S. (1982). *Probability and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam.
Kalmbach, G. (1983). *Orthomodular Lattices*, Academic Press, New York.
Kolmogorov, A. N. (1933). *Grundbegriffe der Wahrscheinlickhkeitsrechnung*, Berlin.
Kôpka, F. (1992). *Tatra Mountains Mathematical Publications*, **1**, 83–87.
Kôpka, F., and Chovanec, F. (1994). *Mathematica Slovaca*, **44**, 21–34.
Lahti, P., and Maczyński, M. J. (1992). *Journal of Mathematical Physics*, **33**, 4133–4138.
Luczak, A. (n.d.). Instruments on von Neumann algebras, Preprint, Lodz University.
Ludwig, G. (1983). *Foundations of Quantum Mechanics*, Vol. 1, Springer-Verlag, New York.
Mackey, G. W. (1963). *Mathematical Foundations of Quantum Mechanics*, Benjamin, New York.
Mushtari, D. K. (1989). *Izvestiya Vuzov matematika*, **8**, 44–52 [in Russian].
Mushtari, D. K., and Matvejchuk, M. S. (1985). *Soviet Mathematics Doklady*, **32**, 36–39.
Navara, M., and Pták, P. (n.d.). Difference posets and orthoalgebras, submitted.
Pták, P. (1986). *Demonstratio Mathematica*, **19**, 349–357.
Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.
Pulmannová, S. (1993*a*). *Tatra Mountains Mathematical Publications*, **3**, 89–100.
Pulmannová, S. (1993*b*). *Reports on Mathematical Physics*, **32**, 235–250.
Pulmannová, S. (1994). Quantum measurements and quantum logics, Preprint SAV. *Journal of Mathematical Physics*, to appear.
Randall, C., and Foulis, D. (1979). New definitions and theorems, University of Massachusetts Mimeographed Notes, Amherst, Massachusetts, Autumn 1979.
Randall, C., and Foulis, D. (1981). Empirical logic and tensor products, in *Interpretations and Foundations of Quantum Theory*, H. Neumann, ed., Vol. 5, Wissenscaftsverlag, Bibliographisches Institut, Mannheim, pp. 9–20.
Sikorski, R. (1964). *Boolean Algebras*, Springer-Verlag, Berlin.
Varadarajan, V. S. (1985). *Geometry of Quantum Theory*, Springer-Verlag, Berlin.
Von Neumann, J. (1932). *Mathematische Grundlagen der Quantenmechanik*, Springer-Verlag, Berlin.